Consider heat conduction in some bulk space $V$ with a boundary $\partial V$. Also consider an infinitesimal space in that bulk called $dV$. Let $u(x, y, z, t)$ represent the temperature in $V$ at any time $t$. Let $E = c\rho u$ where $c$ is the specific heat and $\rho$ is the mass density of the bulk, be the total energy in $dV$.

There are some fundamental laws that will lead us to the heat equation:

<table>
<thead>
<tr>
<th>Fourier heat conduction laws:</th>
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<tbody>
<tr>
<td>(1) If the temperature in a region is constant, there is no heat transfer in that region.</td>
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<tr>
<td>(2) Heat always flows from hot to cold.</td>
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<tr>
<td>(3) The greater the difference between temperatures at two points the faster the flow of heat from one point to the other.</td>
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<tr>
<td>(4) The flow of heat is material dependent.</td>
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</tbody>
</table>

All these laws can be summarized into one equation

$$
\phi(x, y, z, t) = -K_0 \nabla u(x, y, z, t)
$$

(1)

Now we can form a word equation:

(Rate of change of heat) = (Heat flowing into $dV$ per unit time) + (Heat generated in $dV$ per unit time)

(2)

The first statement is the rate of change of the total energy $E$. The second is the flux at $\partial V$ in the normal direction. The third is additional heat being generated in $dV$. For the third statement lets called the additional heat $Q$. This gives us the equation

$$
\frac{\partial}{\partial t} \iiint_V c\rho u \, dV = -\iiint_{\partial V} \phi \cdot n \, dS + \iiint_V Q \, dV
$$

(3)

And using divergence theorem we get

$$
\iiint_{\partial V} \phi \cdot n \, dS = \iiint_V \nabla \cdot \phi \, dV = \iiint_V \nabla \cdot (-K_0 \nabla u) \, dV = K_0 \iiint_V \nabla^2 u \, dV
$$

therefore, the equation becomes

$$
\frac{\partial}{\partial t} \iiint_V c\rho u \, dV = \iiint_V c\rho \frac{\partial}{\partial t} u \, dV = K_0 \iiint_V \nabla^2 u \, dV + \iiint_V Q \, dV \Rightarrow c\rho \frac{\partial u}{\partial t} = K_0 \nabla^2 u + Q.
$$

(4)

If we consider the case $Q = 0$; i.e., no external heat being generated, and if we divide through by $c\rho$, then we get the simplest form of the heat equation

$$
\frac{\partial u}{\partial t} = K \nabla^2 u
$$

(5)

where $K$ is called the thermal diffusivity. In 1-D this is,

$$
\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}
$$

(6)
Heat equation examples. Consider the heat equation with a generic initial condition,
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}; \quad u(x, 0) = f(x). \] (7)
with the following boundary conditions

Ex: \( u(0, t) = u(L, t) = 0. \)

Solution: We make the Ansatz, \( u(x, t) = T(t)X(x). \) Then we plug this into our heat equation
\[ u_t = T'(t)X(x), \quad u_{xx} = T(t)X''(x) \Rightarrow T'X = kTX'' \Rightarrow \frac{T'}{kT} = \frac{X''}{X}. \]
Since the LHS is a function of \( t \) alone, and the RHS is a function of \( x \) alone, and since they are equal, they must equal a constant. Let's call it \( -\lambda^2 \). Then we have
\[ \frac{T'}{kT} = \frac{X''}{X} = -\lambda^2. \] (8)
Notice that I call this from the get go because in our Sturm-Liouville problems the negative eigenvalue case always gave us trivial solutions. Here we bypass that by automatically assuming a positive eigenvalue \( \lambda^2 \). Now we must solve the two differential equations.

The \( T \) equation is the easiest to solve
\[ \frac{T'}{kT} = -\lambda^2 \Rightarrow T' = -k\lambda^2 T \Rightarrow \frac{dT}{dt} = -k\lambda^2 dt \Rightarrow \int \frac{dT}{T} = \int -k\lambda^2 dt \Rightarrow \ln T = -k\lambda^2 t \Rightarrow T = e^{-k\lambda^2 t} \]
Notice that we don’t include the constant in front of the exponential, and that is because the \( X \) equation will have constants, and we would simply by multiplying constants to reduce it to one constant anyway, so I choose to leave it out from the beginning. You don’t have to though.

Now, we solve the \( X \) equation by recalling our Sturm-Liouville problems
\[ \frac{X''}{X} = -\lambda^2 \Rightarrow X'' + \lambda^2 X = 0 \Rightarrow X = A \cos \lambda x + B \sin \lambda x \text{ for } \lambda \neq 0 \text{ and } X = c_1 x + c_2 \text{ for } \lambda = 0. \]
If we look at the \( \lambda = 0 \) case we have \( X(0) = c_2 = 0 \) and \( X(L) = Lc_1 = 0, \) so \( X \equiv 0. \)
Now we look at the \( \lambda \neq 0 \) case. \( X(0) = A = 0 \) and
\[ X(L) = X(L) = B \sin \lambda x = 0 \Rightarrow \lambda = \frac{n\pi}{L} \Rightarrow X_n = B_n \sin \frac{n\pi}{L} x \text{ and } T_n = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \]
Next we combine the \( T \) and \( X \) solutions to get the general solutions,
\[ u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \] (9)
And we can solve for the constants using the principles from Fourier series with the initial condition. Since this is a Fourier sine series we have
\[ u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x) \Rightarrow B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \]
Then our full solution is
\[ u(x, t) = \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \] (10)
Ex: \( u_x(0, t) = u_x(L, t) = 0. \)

Solution: We know from the first example that \( T = e^{-k\lambda^2 t}. \)
For the \( X \) equation we need to look at our two cases. For \( \lambda = 0 \) we have \( X = c_1 x + c_2, \) and \( X'(x) = c_1, \) so for both boundaries \( X'(0) = c_1 = X'(L). \) These leaves us with a constant \( X = c_2. \)

For the \( \lambda \neq 0 \) case we have
\[ X = A \cos \lambda x + B \sin \lambda x \Rightarrow X' = -\lambda A \sin \lambda x + \lambda B \cos \lambda x \]
Then we get \( X'(0) = \lambda B = 0 \) and
\[ X'(L) = -\lambda A \sin \lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L} \Rightarrow X_n = A_n \cos \frac{n\pi x}{L} \text{ and } T_n = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \]
Next we combine the $T$ and $X$ solutions to get our general solution

$$u(x, t) = c_2 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-k \left( \frac{n \pi}{L} \right)^2 t}$$  \hspace{1cm} (11)$$

Now we find our coefficients by invoking the initial condition and using Fourier Series

$$u(x, 0) = c_2 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} = f(x)$$

This gives us

$$c_2 = \frac{1}{L} \int_0^L f(x) dx$$

and

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

Combining everything we get the full solution

$$u(x, t) = \frac{1}{L} \int_0^L f(x) dx + \sum_{n=1}^{\infty} \frac{2}{L} A_n \cos \frac{n\pi x}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$  \hspace{1cm} (12)$$

Ex: Now lets think of heat transfer in a circle. If we go around in one direction we hit $x = -L$ and in the other direction $x = L$, but these are the same point. So we get the following boundary conditions

$$u(-L, t) = u(L, t), \quad u_x(-L, t) = u_x(L, t)$$  \hspace{1cm} (13)$$

**Solution:** We know from the previous two problems that our solutions will be

$$T = e^{-k\lambda^2 t}$$

$$X = c_1 x + c_2$$  for $\lambda = 0$

$$X = A \cos \lambda x + B \sin \lambda x$$  for $\lambda \neq 0$

For $\lambda = 0$, $X(L) = c_1 L + c_2$ and $X(-L) = -c_1 L + c_2$, so $c_1 = 0$. And the derivative is trivially satisfied.

For $\lambda \neq 0$,

$$X(L) = X(-L) \Rightarrow A \cos \lambda L + B \sin \lambda L = A \cos \lambda L - B \sin \lambda L \Rightarrow \sin \lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L}$$

And

$$X'(L) = X'(-L) \Rightarrow -\lambda A \sin \lambda L + \lambda B \cos \lambda L = \lambda A \sin \lambda L + \lambda B \cos \lambda L \Rightarrow \sin \lambda L = 0$$

But we already showed this. So, we need to keep both coefficients. Then our solution for $X$, which as we saw in previous conditions (for the heat equation) is just the initial condition of the general solution, is

$$X = c_2 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} = u(x, 0) = f(x)$$  \hspace{1cm} (14)$$

Now we use Fourier series to solve for the coefficients,

$$c_2 = \frac{1}{L} \int_0^L f(x) dx$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Putting everything back into the general solution gives us

$$u(x, t) = \frac{1}{L} \int_0^L f(x) dx + \frac{2}{L} \sum_{n=1}^{\infty} \cos \frac{n\pi x}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx + \sin \frac{n\pi x}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$  \hspace{1cm} (15)$$
Nonhomogeneous heat conduction examples.

Ex: Consider the following nonhomogeneous boundary condition problem

\[
\frac{\partial u}{\partial t} = k u_{xx}; \quad u(0,t) = A, \quad u(L,t) = B; \quad u(x,0) = f(x).
\]  

(16)

We first look for the easiest solution: the equilibrium temperature. What does equilibrium mean? We solve the problem

\[
\frac{\partial u_\ast}{\partial t} = 0 \Rightarrow \frac{\partial^2 u_\ast}{\partial x^2} = 0; \quad u_\ast(0) = A, \quad u_\ast(L) = B.
\]

(17)

So, \( u_\ast = c_1 x + c_2 \), and \( u_\ast(0) = c_2 = A \), \( u_\ast(L) = c_1 L + A = B \), then our equilibrium solution is

\[ u_\ast = \frac{B-A}{L} x + A. \]

(18)

Obviously, this does not solve the problem, but it does allow us to make a change of variables that makes the B.C.'s homogeneous. Let \( v(x,t) = u(x,t) - u_\ast(x) \). Taking a time derivative kills \( u_\ast \) and taking two spatial derivatives also kills \( u_\ast \), so we get

\[
v_t = kv_{xx}; \quad v(0,t) = v(L,t) = 0; \quad v(x,0) = f(x) - u_\ast = f(x) - \frac{B-A}{L} x + A
\]

(19)

We know

\[
v(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t},
\]

then

\[
v(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = f(x) - \frac{B-A}{L} x + A
\]

\[
\Rightarrow A_n = \frac{2}{L} \int_0^L (f(x) - \frac{B-A}{L} x + A) \sin \frac{n\pi x}{L} dx
\]

which gives us

\[
u(x,t) = \frac{B-A}{L} x + A + \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}
\]

(20)

Ex: Now lets look at an example where the PDE itself is nonhomogeneous

\[
\frac{\partial u}{\partial t} = k u_{xx} + Q; \quad u(0,t) = A, \quad u(L,t) = B; \quad u(x,0) = f(x).
\]  

(21)

Then \( u_{xx} = -\frac{Q}{k} \Rightarrow u_\ast = -\frac{Q x^2}{2k} + c_1 x + c_2 \). Plugging in the BCs gives us \( u_\ast(0) = c_1 = A \) and

\[ u_\ast(L) = -\frac{Q}{2k} L^2 + c_1 L + A = B \Rightarrow c_1 = \frac{1}{L} \left[ B - A + \frac{Q}{2k} L^2 \right] \Rightarrow u_\ast = -\frac{Q}{2k} x^2 + \frac{x}{L} \left[ B - A + \frac{Q}{2k} L^2 \right] + A
\]

(22)

Letting \( v(x,t) = u(x,t) - u_\ast(x) \) gives us our homogenized equation.

4) \[
\frac{\partial u}{\partial t} = k u_{xx}; \quad u(0,t) = u_0, \quad u(1,t) = u_1; \quad u(x,0) = f(x)
\]

(23)

Solution: \( u_{xx} = 0 \Rightarrow u_\ast = c_1 x + c_2 \), so \( u_\ast(0) = c_2 = u_0 \) and \( u_\ast(1) = c_1 + u_0 = u_1 \Rightarrow c_1 = u_1 - u_0 \), then our equilibrium solution is \( u_\ast = (u_1 - u_0)x + u_0 \). Letting \( v = u - u_\ast \) gives us our homogenized equation.