Cross-sectional area $A(x)$: $V = \int_a^b A(x)dx$ (1)

Disk about x-axis: $V = \int_a^b \pi R(x)^2dx$ (2)
Disk about y-axis: $V = \int_a^b \pi R(y)^2dy$ (3)

Washers about x-axis: $V = \int_a^b \pi [R(x)^2 - r(x)^2]dx$ (4)
Washers about y-axis: $V = \int_a^b \pi [R(y)^2 - r(y)^2]dy$ (5)

Cylindrical Shells about y-axis: $V = \int_a^b 2\pi xf(x)dx$ (6)
Cylindrical Shells about x-axis: $V = \int_a^b 2\pi yf(y)dy$ (7)

Integration by parts: $\int u dv = uv - \int v du$. (8)
Strategies for $\int \sin^m x \cos^n x \, dx$.

1. If the power of the cosine term is odd (i.e. $n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$,

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x (\cos^2 x)^k \cos x \, dx$$
$$= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx. \quad (9)$$

Then substitute $u = \sin x \Rightarrow du = \cos x$.

2. If the power of the sine term is odd (i.e. $m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$,

$$\int \sin^{2k+1} \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \, dx = \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx. \quad (10)$$

Then substitute $u = \cos x \Rightarrow du = -\sin x$.

3. If the powers of both sine and cosine are even, use the double-angle formulas:

$$\sin^2 x = \frac{1}{2} (1 - \cos 2x) \quad \cos^2 x = \frac{1}{2} (1 + \cos 2x) \quad \sin x \cos x = \frac{1}{2} \sin 2x.$$

Strategies for $\int \tan^m x \sec^n x \, dx$.

1. If the power of the secant term is even (i.e. $n = 2k$, $k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$,

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx$$
$$= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \, dx. \quad (11)$$

Then substitute $u = \tan x \Rightarrow du = \sec^2 x \, dx$.

2. If the power of the tangent term is odd (i.e. $m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$,

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx$$
$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx. \quad (12)$$

Then substitute $u = \sec x \Rightarrow du = \sec x \tan x \, dx$.
**Useful Integrals.**

These integrals are also pretty easy to derive if you forget them,

\[
\int \tan x \, dx = -\ln |\cos x| + C = \ln |\sec x| + C. \tag{13}
\]

\[
\int \sec x \, dx = \ln |\sec x + \tan x| + C. \tag{14}
\]

**Strategies for trig-sub.**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Substitution</th>
<th>Identity</th>
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</thead>
<tbody>
<tr>
<td>(\sqrt{a^2 - x^2})</td>
<td>(x = a \sin \theta, -\pi/2 \leq \theta \leq \pi/2)</td>
<td>(1 - \sin^2 \theta = \cos^2 \theta)</td>
</tr>
<tr>
<td>(\sqrt{a^2 + x^2})</td>
<td>(x = a \tan \theta, -\pi/2 &lt; \theta &lt; \pi/2)</td>
<td>(1 + \tan^2 \theta = \sec^2 \theta)</td>
</tr>
<tr>
<td>(\sqrt{x^2 - a^2})</td>
<td>(x = a \sec \theta, 0 \leq \theta &lt; \pi/2, \pi &lt; \theta &lt; 3\pi/2)</td>
<td>(\sec^2 \theta - 1 = \tan^2 \theta)</td>
</tr>
</tbody>
</table>

**Midpoint rule:**

\[
\int_a^b f(x) \, dx \approx \Delta x [f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]; \tag{15}
\]

\[x_i^* = \frac{1}{2}(x_i + x_{i+1}), \Delta x = \frac{b-a}{n}\]

Where \(n\) is the number of intervals or equivalently the number of “steps”.

Error bound: \(|E_M| \leq \frac{K(b-a)^3}{24n^2}; |f''(\xi)| \leq K, \xi \in [a,b].\) \tag{16}

Where \(|f''(\xi)|\) is just the maximum of the second derivative in \([a,b]\).

**Trapezoid rule:**

\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]; \tag{17}
\]

\[\Delta x = \frac{b-a}{n}, x_i = a + i\Delta x.\]

Where \(n\) is the number of intervals or equivalently the number of “steps”.

Error bound: \(|E_T| \leq \frac{K(b-a)^3}{12n^2}; |f''(\xi)| \leq K, \xi \in [a,b].\) \tag{18}

Where \(|f''(\xi)|\) is just the maximum of the second derivative in \([a,b]\).
Simpson’s rule:
\[
\int_a^b f(x)\,dx \approx \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)];
\]  
(19)

\[\Delta x = \frac{b-a}{n}, \quad n \geq 4 \quad \text{and} \quad n \text{ must be even.}\]

Where \(n\) is the number of intervals or equivalently the number of “steps”.

Error bound: \(|E_s| \leq \frac{K(b-a)^5}{180n^4}; \quad |f^{(4)}(\xi)| \leq K, \; \xi \in [a,b].\)  
(20)

Where \(|f^{(4)}(\xi)|\) is just the maximum of the fourth derivative in \([a,b]\).

Partial fractions.

Case 1.

Suppose \(Q\) is a product of distinct linear factors, i.e. \(Q = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)\). Then,

\[
P(x) = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}.
\]  
(21)

Case 2.

Suppose \(Q\) is a product of linear factors, some of which are repeated. Then, the repeated factors are of this form

\[
P(x) = \frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_r}{(ax + b)^r}.
\]  
(22)

Case 3.

Suppose \(Q\) is a product of quadratic factors with no repeats, i.e. \(Q = (a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_kx^2 + b_kx + c_k)\). Then,

\[
P(x) = \frac{A_1}{a_1x^2 + b_1x + c_1} + \frac{A_2}{a_2x^2 + b_2x + c_2} + \cdots + \frac{A_k}{a_kx^2 + b_kx + c_k}.
\]  
(23)

Case 4.

Suppose \(Q\) is product of factors that include repeated quadratic factors. Then the repeated quadratic factors will be of the form,

\[
P(x) = \frac{A_1x + B_1}{a_1x^2 + bx + c} + \frac{A_2x + B_2}{a_2x^2 + bx + c} + \cdots + \frac{A_rx + B_r}{a_rx^2 + bx + c}.
\]  
(24)
Improper Integrals.

Case 1: Infinite Intervals

a) If \( \int_a^t f(x) \, dx \) exists for all \( t \geq a \), then \( \int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx \).

b) If \( \int_t^b f(x) \, dx \) exists for all \( t \leq b \), then \( \int_{-\infty}^b f(x) \, dx = \lim_{t \to -\infty} \int_t^b f(x) \, dx \).

Definition 1. If \( \int_a^\infty f(x) \, dx \) and \( \int_{-\infty}^b f(x) \, dx \) are convergent if the limit exists, and divergent if the limit does not exist.

c) If \( \int_a^\infty f(x) \, dx \) and \( \int_{-\infty}^a f(x) \, dx \) are convergent, \( \int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^a f(x) \, dx + \int_a^\infty f(x) \, dx \).

Case 2: Integrands with Discontinuities.

a) If \( f \) is continuous in \([a, b)\) and discontinuous at \( x = b \), then \( \int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx \).

b) If \( f \) is continuous in \((a, b]\) and discontinuous at \( x = a \), then \( \int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx \).

Definition 2. The integral \( \int_a^b f(x) \, dx \) is said to be convergent if the limit exists, and divergent if the limit does not exist.

c) If \( f \) has a discontinuity at \( c \in [a, b] \) and \( \int_a^c f(x) \, dx \) and \( \int_c^b f(x) \, dx \) both converge, then
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
\]

Taylor series:
\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \quad (25)
\]

Remainder:
\[
|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}.
\quad (26)
\]
Common Taylor Series

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \]  
(27)

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \]  
(28)

\[ \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \]  
(29)

\[ \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots . \]  
(30)

Parametric derivative:

\[ y' = \frac{dy}{dx} = \frac{dy}{dt} \frac{dx}{dt} = \frac{g'(t)}{f'(t)}; \quad \frac{dx}{dt} \neq 0. \]  
(31)

Second derivative:

\[ \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} \frac{dx}{dt} \frac{d^2 x}{dt^2} = \frac{g'(t)/f'(t)}{f'(t)^2} = \frac{g'(t)}{f'(t)^2}; \quad \frac{dx}{dt} \neq 0. \]  
(32)

Arc Length:

\[ L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_{\alpha}^{\beta} \sqrt{f'(t)^2 + g'(t)^2} \, dt. \]  
(33)

Surface Area about x-axis:

\[ SA = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]  
(34)

Surface Area about y-axis:

\[ SA = \int_{\alpha}^{\beta} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]  
(35)

Polar Coordinates:

\[ x = r \cos \theta, \quad y = r \sin \theta; \quad r^2 = x^2 + y^2, \quad \theta = \tan^{-1}(y/x) \]  
(36)

Area of a wedge:

\[ A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta. \]  
(37)

Polar Arc Length:

\[ L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta. \]  
(38)