FALL 2015 SOLUTIONS

(1) (a) We need to change this to the following series,
\[ 4 \sum_{n=0}^{\infty} \frac{4^n}{(n+2)^n} \]

Then we use Root test,
\[ \lim_{n \to \infty} \sqrt[n]{\frac{4^n}{(n+2)^n}} = \lim_{n \to \infty} \frac{4^n}{n+2} = 0 < 1 \]

Therefore, by root test the sum converges.

(b) Here we can use direct comparison,
\[ \cos^2 \frac{n}{n^2+1} \leq \frac{2}{n^2} \]

and \( \sum_{n=1}^{\infty} \frac{2}{n^2} \) converges since \( p > 1 \), therefore by DCT the series converges.

(2) (a) We use LCT with \( \frac{\sin(\pi/n)}{\pi/n} \),
\[ \lim_{n \to \infty} \frac{\sin(\pi/n)}{\pi/n} = 1. \]

and \( \sum_{n=1}^{\infty} \frac{\pi/n}{n} \) diverges since \( p \leq 1 \), therefore by LCT the series diverges.

(b) We use LCT with \( \frac{1}{(2^n/n)^n} \),
\[ \lim_{n \to \infty} \frac{(n+2^{-n})/(n2^n-1)}{1/2^n} = \lim_{n \to \infty} \frac{n2^n+1}{n2^n-1} = 1. \]

and \( \sum_{n=1}^{\infty} (1/2)^n \) converges since \( 1/2 < 1 \), hence the series converges by LCT.

(3) (a) Here we use Ratio test,
\[ \lim_{n \to \infty} \left| \frac{\ln(n+1)(3/4)^{n+1}}{\ln(n)(3/4)^n} \right| = \lim_{n \to \infty} \frac{3}{4} \frac{1/(n+1)}{1/n} = \lim_{n \to \infty} \frac{3}{4} \frac{n}{n+1} = \frac{3}{4} < 1. \]

Therefore, by the ratio test the series converges absolutely.

(b) Here we just take the limit,
\[ \lim_{n \to \infty} \frac{1}{1+e^{1/n}} = \frac{1}{2} \neq 0 \]

Therefore, it diverges.

(4) Lets first try absolute convergence using LCT with \( 1/\sqrt{n} \),
\[ \lim_{n \to \infty} \frac{n/\sqrt{n^3+3}}{1/\sqrt{n}} = \lim_{n \to \infty} \frac{n^{3/2}}{\sqrt{n^3+3}} = 1. \]

and \( \sum_{n=1}^{\infty} 1/\sqrt{n} \) diverges since \( p \leq 1 \). Now, lets do AST, \( \lim_{n \to \infty} n/\sqrt{n^3+3} = 0 \), so we proceed with showing decreasing,
\[ \left( \frac{n}{\sqrt{n^3+3}} \right)' = \frac{\sqrt{n^3+3} - \frac{1}{2}(n^3+3)^{-1/2}3n^3}{n^3+3} = \frac{n^3+3 - \frac{3n^3}{2}}{(n^3+3)^{3/2}}. \]

Notice, \( 3 - n^3/2 < 0 \) for all \( n \geq 1 \), so the series is conditionally convergent by AST.
(5) As usual we do ratio test, 
\[
\lim_{n \to \infty} \left| \frac{(x-1)^{n+1}}{n!} \cdot \frac{(n-1)!}{(x-1)^n} \right| = \lim_{n \to \infty} \frac{1}{n} |x-1| = 0
\]
This shows that \( R = \infty \) and \( x \in (-\infty, \infty) \).

(6) Again we do ratio test, 
\[
\lim_{n \to \infty} \left| \frac{(x+2)^{n+1}}{(n+3)^{3n+1}} \cdot \frac{n^{3n}}{(x+2)^n} \right| = \lim_{n \to \infty} \frac{n}{n+1} \left| \frac{x+2}{3} \right| = \frac{|x+2|}{3} < 1 \Rightarrow |x+2| < 3.
\]
This shows, \( R = 3 \), and \( x \in (-5, 1) \). Now let’s test the end points. For \( x = 5 \), \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges since \( p \leq 1 \). For \( x = 1 \), \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges conditionally by AST, since \( \lim_{n \to \infty} \frac{1}{n} = 0 \) and \( \frac{1}{n} > \frac{1}{(n+1)} \). Therefore, \( x \in (-5, 1) \).

(7) The Taylor series of \( \cos x \) about \( x = 0 \) is, 
\[
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \Rightarrow \cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = 1 - 2x^2 + \frac{2}{3} x^4 + \cdots
\]
Therefore, 
\[
(1 + x^2) \cos(2x) \approx (1 + x^2) \left( 1 - 2x^2 + \frac{2}{3} x^4 \right) = 1 - x^2 - \frac{4}{3} x^4.
\]

(8) Here we just take derivatives and compute, with \( f(\pi/4) = 1 \).
\[
f'(x) = 2 \cos(2x) \bigg|_{x=\pi/4} = 0, \quad f''(x) = -4 \sin(2x) \bigg|_{x=\pi/4} = -4
\]
\[
f''(x) = -8 \cos(2x) \bigg|_{x=\pi/4} = 0, \quad f^{(4)}(x) = 16 \sin(2x) \bigg|_{x=\pi/4} = 16
\]
\[
\Rightarrow f(x) \approx 1 - 2(x - \pi/4)^2 + \frac{2}{3} (x - \pi/4)^4.
\]

(9) (a) Again we take derivatives and compute, with \( f(1) = e \).
\[
f'(x) = e^x + xe^x \bigg|_{x=1} = 2e, \quad f''(x) = 2e^x + xe^x \bigg|_{x=1} = 3e
\]
\[
\Rightarrow f(x) \approx e + 2e(x - 1) + \frac{3}{2} e(x - 1)^2.
\]
(b) Here we take the third derivative and evaluate it at \( x = 2 \) and evaluate \( |x - 1|^3 \) at \( x = 2 \), 
\[
f'''(x) = 3e^x + xe^x \bigg|_{x=2} = 5e^2 \Rightarrow |R_2| = \frac{5e^2}{6}
\]