Exam III Fall 2016:

(1) Converges by GST since $|r| = 1/3 < 1$ and

$$\sum_{n=2}^{\infty} 5 \left(\frac{-1}{3}\right)^n = \sum_{n=0}^{\infty} 5 \left(\frac{-1}{3}\right)^{n+2} = \sum_{n=0}^{\infty} \frac{5}{9} \left(\frac{-1}{3}\right)^n = \frac{5/9}{1 + 1/3} = \frac{3 \cdot 5}{12}$$

(2) (a) Converges by GST since $|r| = 3/4 < 1$ because

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{2^{2n}} = \sum_{n=0}^{\infty} 3 \frac{3^n}{4^n} = 3 \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{3}{1 - 3/4} = 12.$$ (b) This diverges by the nth term test since

$$\lim_{n \to \infty} \frac{n^2}{(n+2)(n-2)} = \lim_{n \to \infty} \frac{1}{(1+2/n)(1-2/n)} = 1 \neq 0$$

(3) (a) We compare to $1/n^{1/3}$,

$$\lim_{n \to \infty} \frac{2n^7 + n}{1/n^{1/3}} = \lim_{n \to \infty} \frac{2n^{1/3}}{\sqrt[n]{n^7 + n}} = \lim_{n \to \infty} \frac{2}{\sqrt[n]{1 + 1/n^6}} = 2\sqrt[3]{2}$$

$$\sum_{n=1}^{\infty} 1/n^{1/3}$$ diverges by p-test since $p < 1$, so the original series also diverges by LCT.

(b) Notice $n/(e^{-n} + n^2) \leq 1/n^2$ and $\sum_{n=1}^{\infty} 1/n^2$ converges by p-test since $p > 1$, so the original series converges by DCT.

(4) This is one of the few problems where we have to use integral test,

$$\lim_{t \to \infty} \int_2^t \frac{dx}{x \sqrt{\ln x}} = \lim_{t \to \infty} \int_2^t \frac{du}{\sqrt{u}} = \lim_{t \to \infty} 2u^{1/2} \bigg|_2^t = \lim_{t \to \infty} 2\sqrt{\ln t} - 2\sqrt{\ln 2} = \infty$$

So the absolute value diverges by integral test. But now we have to test for conditional convergence of the original series by AST. Notice $\lim_{n \to \infty} 1/n \sqrt{n} = 0\sqrt{1}$ and $1/((n+1)\sqrt{(n+1)}) \leq 1/n \sqrt{n\sqrt{1}}$, therefore the original series converges conditionally by AST.

(5) This converges absolutely since $|f(n)| = f(n) > 0$.

(6) Here we use ratio test,

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(x-1)^{n+1}}{(n+1)(n+1)^{3}+1} \frac{n^{3n}}{(x-1)^n} = \lim_{n \to \infty} \frac{1}{3} \left( \frac{n}{n+1} \right) |x-1| = \lim_{n \to \infty} \frac{1}{3} \left( \frac{1}{1+1/n} \right) |x-1| = \frac{1}{3} |x-1| < 1$$

Hence $|x-1| < 3 = R$ and the interval of absolute convergence is $-2 < x < 4$. If $x = 4$,

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^{3n}} = \sum_{n=1}^{\infty} \frac{1}{n}$$ diverges by p-test since $p = 1$. If $x = -2$,

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^{3n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ converges conditionally by AST since $\lim_{n \to \infty} 1/n = 0\sqrt{1}$ and $1/(n+1) \leq 1/n$ and $\sum_{n=1}^{\infty} |(-1)^n/n|$ diverges as shown above. Therefore the radius of convergence is $R = 3$ and the interval of convergence is $x \in [-2, 4)$. 

(7) (a) We first find the polynomial of order 3. \( f(3) = 0, f'(3) = (x - 2)^{-1}\big|_{x=3} = 1, \) 
\( f''(3) = -(x - 2)^{-2}\big|_{x=3} = -1, \) and \( f'''(3) = 2(x - 2)^{-3}\big|_{x=3} = 2, \) so 
\[ P_3(x) = (x - 3) - \frac{1}{2}(x - 3)^2 + \frac{1}{3}(x - 3)^3. \]

(b) We look for the pattern in the nth derivative, \( f^{(n)}(3) = (-1)^{n+1}(n - 1)! \), so we get 
\[ f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!}(x - 3)^n \]

(8) (a) Here we can see that we get a factor of \(-1/2\) at each derivative so, \( f^{(1/2)}(1) = (-1/2)^n e^{-1/2} \) then our Taylor series is 
\[ f(x) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n e^{-1/2} (x - 1)^n \]

(b) For the radius of convergence we use ratio test 
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1/2)^{n+1}e^{-1/2}(x - 1)^{n+1}/(n + 1)!}{(-1/2)^n e^{-1/2}(x - 1)^n/n!} \right| = \lim_{n \to \infty} |x - 1| = 0 \]

Therefore the radius of convergence is \( R = \infty \) and the interval of convergence is \( x \in (-\infty, \infty) \)

(9) We use the common Taylor series, 
\[ \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} \]

(10) Now we integrate the above, 
\[ \int \cos(x^2) \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!(4n + 1)} + C \]

(11) Now we need to find the remainder in order to get the proper approximation error, 
\[ |R_n(x)| \leq \frac{x^{4n+5}}{(2n + 2)!(4n + 5)} \Rightarrow \max(|R_n(x)|) \leq \frac{1}{(2n + 2)!(4n + 5)} \leq \frac{1}{100} \]

Since this cannot be solved easily, we just plug in values of \( n \) until we get the sufficient error. 
\( n = 0 \Rightarrow |\text{Error}| \leq 1/10 \) and \( n = 1 \Rightarrow |\text{Error}| \leq 1/(24 \cdot 9) < 1/100. \) So, \( n = 1 \) works, i.e. 
\[ \int \cos(x^2) \, dx \approx x - \frac{x^5}{10} \]