We briefly went over a bunch of topics that were covered in Calc II.

Distance formula in 3-D: If \( P_1 = (x_1, y_1, z_1) \) and \( P_2 = (x_2, y_2, z_2) \),
\[
d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.
\]

Know how to add vectors, multiply vectors by a scalar, find the magnitude of vectors, the different forms of writing a vector (e.g., column vector, using \( \hat{i} = (1, 0, 0), \hat{j} = (0, 1, 0), \hat{k} = (0, 0, 1) \)), and how to find a unit vector.

We sketched some simple planes, such as \( z = 3 \), \( y = 5 \), and \( y = x \).

Ex: The distance between the points \( P = (2, -1, 7) \) and \( Q = (1, -3, 5) \) is \( d(P, Q) = 3 \).

Equation of sphere: By definition a sphere is the set of all points \( P = (x, y, z) \) whose distance from the center \( C = (h, k, l) \) is \( r \), so \( d(P, C) = r \Rightarrow d(P, C)^2 = r^2 \Rightarrow (x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 \).

Ex: Show that \( x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0 \) is the equation of a sphere.

**Solution:** Complete the square:
\[
(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) = -6 + 4 + 9 + 1
\]
\[
\Rightarrow (x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8,
\]
which means the center is \( C = (2, -3, 1) \) and \( r = 2\sqrt{2} \).

Ex: What region in \( \mathbb{R}^3 \) is represented by \( 1 \leq x^2 + y^2 + z^2 \leq 4; z \leq 0 \)?

**Solution:** In class we sketched this, but here we will forgo the sketch. Just recall that it is a hemisphere in the bottom \( \mathbb{R}^3 \) with a sphere of radius unity hollowed out.

Properties of vectors

| 1 | \( \vec{a} + \vec{b} = \vec{b} + \vec{a} \), |
| 2 | \( \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} \), |
| 3 | \( \vec{a} + \vec{0} = \vec{a} \), |
| 4 | \( \vec{a} + (-\vec{a}) \), |
| 5 | \( c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b} \), |
| 6 | \( (c + d)\vec{a} = c\vec{a} + d\vec{a} \), |
| 7 | \( (cd)\vec{a} = c(d\vec{a}) \), |
| 8 | \( 1\vec{a} = \vec{a} \) |

Dot products Recall that a dot product in \( \mathbb{R}^2 \) works as such
\[
\begin{bmatrix} 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 2 \times 3 + 4 \times (-1) = 2.
\]

In \( \mathbb{R}^3 \) it works the same way
\[
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3
\]
(1)
Properties of dot products

1. \( \vec{a} \cdot \vec{a} = \|\vec{a}\|^2 \),
2. \( \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \),
3. \( \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \),
4. \( (c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) \),
5. \( \vec{0} \cdot \vec{a} = 0 \).

Law of cosines: \( \vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos \theta \)

Ex: If \( \|\vec{a}\| = 4 \) and \( \|\vec{b}\| = 6 \), and \( \theta = \pi/3 \) (angle between \( \vec{a} \) and \( \vec{b} \)), then \( \vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos \theta = 12 \).

Ex: Find the angle between \( \vec{a} = \langle 1, 2, -1 \rangle \) and \( \vec{b} = \langle 5, -3, 2 \rangle \).
   Solution: \( \|\vec{a}\| = 3, \|\vec{b}\| = \sqrt{38}, \vec{a} \cdot \vec{b} = 2 \), then \( \theta = \cos^{-1} \left( \frac{2}{3\sqrt{38}} \right) \).

Two nonzero vectors \( \vec{a} \) and \( \vec{b} \) are orthogonal (perpendicular) if and only if \( \vec{a} \cdot \vec{b} = 0 \). This is due to the law of cosines. When we have two vectors that are perpendicular the angle between them is \( \pi/2 \), and therefore \( \cos \theta = 0 \).

Ex: Show that \( 2\hat{i} + 2\hat{j} - \hat{k} \) is perpendicular to \( 5\hat{i} - 4\hat{j} + 2\hat{k} \).
   Solution: \( \langle 2, 2, -1 \rangle \cdot \langle 5, -4, 2 \rangle = 0 \)

Direction angles and cosines

Suppose \( \alpha, \beta, \gamma \) are angles of a vector \( \vec{a} = \langle a_1, a_2, a_3 \rangle \) from the \( x, y, z \) axes. Then

\[
\cos \alpha = \frac{\vec{a} \cdot \hat{i}}{\|\vec{a}\| \|\hat{i}\|} = \frac{a_1}{\|\vec{a}\|}, \quad \cos \beta = \frac{\vec{a} \cdot \hat{j}}{\|\vec{a}\| \|\hat{j}\|} = \frac{a_2}{\|\vec{a}\|}, \quad \cos \gamma = \frac{\vec{a} \cdot \hat{k}}{\|\vec{a}\| \|\hat{k}\|} = \frac{a_3}{\|\vec{a}\|},
\]

Projections

Suppose we want to find the projection of one vector \( \vec{b} \) onto another vector \( \vec{a} \). That is, find a vector in the direction of \( \vec{a} \) with a magnitude that is equivalent to the magnitude of the component of \( \vec{b} \) in the direction of \( \vec{a} \).

The component of \( \vec{b} \) in the direction of \( \vec{a} \) will be

\[
\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|},
\]
then the projection is simply

\[
\text{proj}_{\vec{a}} \vec{b} = \left( \text{comp}_{\vec{a}} \vec{b} \right) \frac{\vec{a}}{\|\vec{a}\|} = \left( \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|} \right) \frac{\vec{a}}{\|\vec{a}\|}.
\]

Ex: Project \( \vec{b} = \langle 1, 1, 2 \rangle \) onto \( \langle -2, 3, 1 \rangle \)

\[
\text{comp}_{\vec{a}} \vec{b} = \frac{3}{\sqrt{14}} \Rightarrow \text{proj}_{\vec{a}} \vec{b} = \frac{3}{\sqrt{14}} \frac{\vec{a}}{\|\vec{a}\|} = \begin{bmatrix} -3/7 \\ 9/14 \\ 3/14 \end{bmatrix}
\]

Ex: Recall that work is a dot product of force and distance: \( W = Fd \cos \theta = \vec{F} \cdot \vec{d} \).
Cross products

If \( \vec{a} = \langle a_1, a_2, a_3 \rangle \) and \( \vec{b} = \langle b_1, b_2, b_3 \rangle \), then through cofactor expansion we get

\[
\vec{a} \times \vec{b} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{vmatrix}
= \begin{vmatrix}
a_2 & a_3 \\
b_2 & b_3
\end{vmatrix}\hat{i} - \begin{vmatrix}
a_1 & a_3 \\
b_1 & b_3
\end{vmatrix}\hat{j} + \begin{vmatrix}
a_1 & a_2 \\
b_1 & b_2
\end{vmatrix}\hat{k}
= (a_2 b_3 - a_3 b_2)\hat{i} - (a_1 b_3 - a_3 b_1)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}.
\]

Ex: If \( \vec{a} = \langle 1, 3, 4 \rangle \) and \( \vec{b} = \langle 2, 7, -5 \rangle \), \( \vec{a} \times \vec{b} = -43\hat{i} + 13\hat{j} + \hat{k} \).

Ex: \( \vec{a} \times \vec{a} = 0 \).

Recall that the vector from the cross product \( \vec{a} \times \vec{b} \) is orthogonal to both \( \vec{a} \) and \( \vec{b} \). This gives us the right hand rule that you would have learned in physics. And we can prove this by using the dot product: \( (\vec{a} \times \vec{b}) \cdot \vec{a} = 0 \).

Here are some other consequences from the cross product:

- If \( \theta \) is the angle between \( \vec{a} \) and \( \vec{b} \), \( ||\vec{a} \times \vec{b}|| = ||\vec{a}|| ||\vec{b}|| \sin \theta \).
- Two vectors are parallel if and only if \( \vec{a} \times \vec{b} = 0 \).
- The length of \( \vec{a} \times \vec{b} \) is given by the area of the parallelogram determined by \( \vec{a} \) and \( \vec{b} \).

Properties of cross products

1. \( \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \) (i.e., does not commute),
2. \( c\vec{a} \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b}) \),
3. \( \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \),
4. \( (\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c} \),
5. \( \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{b}) \cdot \vec{c} \),
6. \( \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{b})\vec{c} - (\vec{a} \cdot \vec{c})\vec{b} \).

Another interesting consequence of the cross product is the volume of a parallelepiped. Suppose the vectors \( \vec{a}, \vec{b}, \vec{c} \) make up the sides of the parallelepiped, then the magnitude of the triple product gives us the volume: \( V = ||\vec{a} \cdot (\vec{b} \times \vec{c})|| \).

Ex: Find a vector perpendicular to the plane that passes through the points \( P(1, 4, 6) \), \( Q(-2, 5, -1) \), and \( R(1, -1, 1) \).

Solution: While we may not have the origin in this plane, we may redefine the “origin” to be a point on the plane, lets say \( P \). Then we have two vectors: \( \vec{PQ} = Q - P = \langle -3, 1, -7 \rangle \) and \( \vec{PR} = R - P = \langle 0, -5, -5 \rangle \). Now the cross product gives us the perpendicular vector

\[
\vec{PQ} \times \vec{PR} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-3 & 1 & -7 \\
0 & -5 & -5
\end{vmatrix}
= -40\hat{i} - 15\hat{j} + 15\hat{k}.
\]
Ex: Let’s do the following interesting derivation

\[ \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = b_2 c_3 \hat{i} - b_1 c_3 \hat{j} + b_1 c_2 \hat{k} \]

\[ \Rightarrow \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} a_1 - \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} a_2 + \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} a_3 \]

Physics/Engineering examples.

Ex: A crate is hauled 8m up a ramp under a constant force of 200N applied at an angle of 25° to the ramp. Find the work done.

**Solution:** We sketched a free-body diagram and notice that whether we use basic Trigonometry or law of cosines with the dot product we get the same thing:

\[ W = F \cdot d = \|F\|\|d\| \cos(25^\circ) = 1450 \text{J}. \]

Ex: A force \( F = 3\hat{i} + 4\hat{j} + 5\hat{k} \) moves an object from point \( P(2, 1, 0) \) to point \( Q(4, 6, 2) \). Find the work done.

**Solution:** \( W = F \cdot d = F \cdot \vec{PQ} = 36. \)

Recall that torque can be computed using the cross product,

\[ \tau = r \times F \Rightarrow \|\tau\| = \|r \times F\| = \|r\|\|F\| \sin \theta. \]

Ex: A bolt is tightened by applying a 40N force to a 0.25m wrench at a 75° angle. Find the magnitude of the torque.

**Solution:**

\[ \|\tau\| = \|r \times F\| = \|r\|\|F\| \sin(75^\circ) = (0.25)(40) \sin(75^\circ) = 10 \sin(75^\circ). \]

If the bolt is right-threaded it will go out of the board.
How do we determine a line in \( \mathbb{R}^2 \)? We take a point on the line and find the direction (slope). The easiest point to use is the y-intercept: \( y = mx + b \).

Similarly for \( \mathbb{R}^3 \) suppose we want to determine a line \( L \) that crosses through some point \( P_0(x_0, y_0, z_0) \). We can’t use the y-intercept and the slope because the line may not intersect one of the axes, and we don’t have a slope (recall that slope is rise over run).

However, we can use a point on the line, say \( P_0 \) and find a direction, which is a vector \( \vec{v} \) from the origin that is parallel to the line. In order to find this direction define an arbitrary point on the line: \( P(x, y, z) \). Then \( P_0\vec{P} = \langle P - P_0 \rangle = \vec{r} - \vec{r}_0 \) gives us the direction. Since they are in the same direction, \( P_0\vec{P} = \vec{r} - \vec{r}_0 = t\vec{v} \) where the scalar \( t \) is the parameter of the line. Then the line is uniquely determined by \( \vec{r} - \vec{r}_0 = t\vec{v} \). Since \( \vec{r} = \langle x, y, z \rangle \) and \( \vec{r}_0 = \langle x_0, y_0, z_0 \rangle \), we can write down the vector equation of the line
\[
\langle x, y, z \rangle = \vec{r} = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle.
\] (2)

Some more algebra gives us the parametric representation of a line
\[
x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.
\] (3)

As \( t \) varies we move up and down the line.