Section 9.7 Quadratic surfaces (continued)

Ex: Sketch $z = 4x^2 + y^2$

**Solution:** What problem do we run into in the $xy$-plane?

Notice that if $z = 0$, then $x = y = 0$, and $z \geq 0$, so let's take some $k \geq 0$. Now if we look at $z = k$ we get the equation $4x^2 + y^2 = k$, which is an ellipse.

Then in the $xz$-plane we get $y = 4x^2$, and similarly in the $yz$-plane we get $z = y^2$; both of which are parabolas. So, this is an elliptic paraboloid.

Ex: Sketch $z = y^2 - x^2$.

**Solution:** Notice that unlike the previous problem, $z$ can be negative and $z = 0$ isn't an issue. So we have three cases, $z = 0 \Rightarrow y = \pm x$, $z = 1 \Rightarrow y^2 - x^2 = 1$, $z = -1 \Rightarrow y^2 - x^2 = -1 \Rightarrow x^2 - y^2 = 1$, which gives us hyperbolas. For this problem it is useful to sketch the trace (plot on the left). Then for the other two directions we have $z = -x^2$ in the $xz$-plane and $z = y^2$ in the $yz$-plane, which are paraboloids. So we have a hyperbolic paraboloid also known as a saddle.
Ex: Sketch $x^2/4 + y^2 - z^2/4 = 1$.

**Solution:** The $xy$-plane gives us $x^2/4 + y^2 = 1$, $xz$-plane: $x^2/4 - z^2/4 = 1$, and $yz$-plane: $y^2 - z^2/4 = 1$. So we have an ellipse in the $xy$-plane and hyperbolas in the other, so this is a Hyperboloid. Notice in the plot, that the hyperboloid is connected, and therefore of one sheet (plot on the left).

Ex: Sketch $4x^2 - y^2 + 2z^2 + 4 = 0$.

**Solution:** In standard form this is $-x^2 + y^2/4 - z^2 = 1$. Then for the $xy$-plane we get $-x^2 + y^2/4 = 1$. This is a hyperbola, but notice that $|y| \geq 2$, otherwise $x$ would be imaginary. In the $xz$-plane we get $x^2 + z^2/2 = 1 - k^2/4$ if we let $y^2 = k^2 \geq 4$, which is an ellipse. Finally, on the $yz$-plane we get $y^2/4 - z^2/2 = 1$, which is a hyperbola. So we get a hyperboloid once again, however since it is not connected this will be of two sheets (plot on the right).

![Graph of a hyperboloid](image1)

Ex: Classify the quadratic surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

**Solution:** Notice that this is not in standard form. Everything is fine except the $x$ portion. If we complete the square we get

$$(x - 3)^2 + 2z^2 - y + 1 = 0.$$ 

So, this has a critical point of $(3, 1, 0)$. By looking at the traces: $z = 0$: $y = (x - 3)^2$ (parabola), $y = k > 1$: $(x - 3)^2 + 2z^2 = k - 1$ (ellipse), and $x = 3$: $y = 2z^2 + 1$ (parabola), we see that it is an elliptic paraboloid.

**Section 10.1 Vector functions**

A vector valued function is a vector where each component is a function: $\vec{r}(t) = (f(t), g(t), h(t)) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$.

Ex: If $\vec{r}(t) = (t^3, \ln(3 - t), \sqrt{t})$, then $f(t) = t^3$, $g(t) = \ln(3 - t)$, and $h(t) = \sqrt{t}$. Notice that $t \in (-\infty, \infty)$ for $f(t)$, $(-\infty, 3)$ for $g(t)$, and $[0, \infty)$ for $h(t)$. So, the domain is $[0, 3)$. 

![Graph of a vector function](image2)
Definition 1. If \( \mathbf{r}(t) = (f(t), g(t), h(t)) \), then
\[
\lim_{t \to t_0} \mathbf{r}(t) = \left( \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \right)
\]
provided the limits of the component functions exist.

Properties of limits

1. Sum: \( \lim_{t \to t_0} (\mathbf{r}_1(t) + \mathbf{r}_2(t)) = \lim_{t \to t_0} \mathbf{r}_1(t) + \lim_{t \to t_0} \mathbf{r}_2(t) \).
2. Scalar multiple: \( \lim_{t \to t_0} (c \mathbf{r}(t)) = c \lim_{t \to t_0} \mathbf{r}(t) \).
3. Dot product: \( \lim_{t \to t_0} (\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)) = (\lim_{t \to t_0} \mathbf{r}_1(t)) \cdot (\lim_{t \to t_0} \mathbf{r}_2(t)) \).
4. Cross product: \( \lim_{t \to t_0} (\mathbf{r}_1(t) \times \mathbf{r}_2(t)) = (\lim_{t \to t_0} \mathbf{r}_1(t)) \times (\lim_{t \to t_0} \mathbf{r}_2(t)) \).

Ex: Find \( \lim_{t \to 0} \mathbf{r}(t) \) where \( \mathbf{r}(t) = (1 + t^3) \mathbf{i} + t e^{-t} \mathbf{j} + \sin t / t \mathbf{k} \).

Solution:
\[
\lim_{t \to 0} \mathbf{r}(t) = \left( \lim_{t \to 0} (1 + t^3), \lim_{t \to 0} t e^{-t}, \lim_{t \to 0} \frac{\sin t}{t} \right)
\]

Just like with limits, we have a definition of continuity.

Definition 2. A vector function \( \mathbf{r} \) is continuous at \( t_0 \) if \( \lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0) \); i.e., it is continuous if its components are continuous.

Ex: Describe the curve defined by \( (1 + t, 2 + 5t, -1 + 6t) \).

Solution: Notice that this is just a line through \((1, 2, -1)\) with direction vector \((1, 5, 6)\).

Ex: Sketch the curve for \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k} \).

Solution: This is a circle in the \( xy \)-plane that moves up the \( z \) direction; i.e., a helix.

Ex: Find the vector equation and parametric equation of the line through \( P(1, 3, -2) \) and \( Q(2, -1, 3) \).

Solution: The initial point is \( P(1, 3, -2) \) and the direction vector is \( \mathbf{v} = (1, -4, 5) \). Then the vector and parametric equations are
\[
\mathbf{r}(t) = \langle 1 + t, 3 - 4t, -2 + 5t \rangle \\
x = 1 + t, \quad y = 3 - 4t, \quad z = -2 + 5t
\]

Ex: Find a vector function that represents the curve of intersection of the cylinder \( x^2 + y^2 = 1 \) and plane \( y + z = 2 \).

Solution: Notice that \( x = \cos t \) and \( y = \sin t \), since the cylinder is just a circle in the \( xy \)-plane. Now we just need a parameterization of \( z \). Since \( z = 2 - y = 2 - \sin t \). Then
\[
\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + (2 - \sin t) \mathbf{k}; 0 \leq t \leq 2\pi.
\]
**Theorem 1.** If \( \vec{r}(t) = \langle f(t), g(t), h(t) \rangle \) and \( f, g, h \) are differentiable, then
\[
\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle.
\] (2)

What does a derivative represent? In 2-D it is the slope of the tangent, so in 3-D it is the direction vector of the tangent line.

**Ex:** (a) Find the derivative of \( \vec{r}(t) = (1 + t^3) \hat{i} + te^{-t} \hat{j} + \sin(2t) \hat{k} \)

**Solution:**
\[
\vec{r}'(t) = 3t^2 \hat{i} + (1 - t)e^{-t} \hat{j} + 2 \cos(2t) \hat{k}.
\]

(b) Find the unit tangent vector at the point where \( t = 0 \).

**Solution:** \( \vec{r}(0) = \hat{i} \) and \( \vec{r}'(0) = \hat{j} + 2 \hat{k} \), so the unit tangent vector at point \((1,0,0)\) is
\[
T(0) = \frac{\vec{r}'(0)}{\|\vec{r}'(0)\|} = \frac{1}{\sqrt{5}} \hat{i} + \frac{2}{\sqrt{5}} \hat{j}
\]

**Ex:** For the curve \( \vec{r}(t) = \sqrt{t} \hat{i} + (2 - t) \hat{j} \), find \( \vec{r}'(t) \) and sketch the position vector \( \vec{r}(1) \) and the tangent vector \( \vec{r}'(1) \).

**Solution:** \( \vec{r}'(t) = (1/2 \sqrt{t}, -1) \), so \( \vec{r}'(1) = (1/2, -1) \). For the sketch, notice that this is a curve on the \( xy \)-plane and \( y = 2 - x^2 \) with \( x \geq 0 \).

**Ex:** Find the parametric equation for the tangent line to the helix \( x = 2 \cos t, y = \sin t, \) and \( z = t \) at point \((0, 1, \pi/2)\).

**Solution:** We first notice that \( t = \pi/2 \), then \( \vec{r}'(t) = (- \sin t, \cos t, 1) \), and \( \vec{r}'(\pi/2) = (-2, 0, 1) \). So, the tangent line is the line through point \((0, 1, \pi/2)\) and parallel to \((-2, 0, 1)\); i.e., the tangent line has the parametric form \( x = -2t, y = 1, \) and \( z = \pi/2 + t \).

**Ex:** Determine whether \( \vec{r}(t) = \langle 1 + t^3, t^2 \rangle \) is smooth (\( \vec{r}'(t) \neq 0 \) and continuous).

**Solution:** First we take the derivative \( \vec{r}'(t) = (3t^2, 2t) \), which is zero at \( t = 0 \), so it is not smooth as it has a cusp at \((1,0)\). But it is smooth at all other points, and hence is called piecewise smooth.

**Ex:** Show that if \( \|\vec{r}(t)\| = c \) where \( c \) is a constant, then \( \vec{r}'(t) \) is orthogonal to \( \vec{r}(t) \) for all \( t \).

**Solution:** Since \( \vec{r}(t) \cdot \vec{r}(t) = \|\vec{r}(t)\|^2 = c^2 \), the derivative is
\[
\frac{d}{dt} [\vec{r}(t) \cdot \vec{r}(t)] = \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 2\vec{r}'(t) \cdot \vec{r}(t) = 0.
\]
Therefore, \( \vec{r}'(t) \) is orthogonal to \( \vec{r}(t) \).

**Theorem 2.**
\[
\int_a^b \vec{r}(t)dt = \left( \int_a^b f(t)dt \right) \hat{i} + \left( \int_a^b g(t)dt \right) \hat{j} + \left( \int_a^b h(t)dt \right) \hat{k}
\] (3)
If $\vec{r}(t) = (2 \cos t)\hat{i} + (\sin t)\hat{j} + (2t)\hat{k}$, then

$$\int \vec{r}(t)dt = (2 \sin t)\hat{i} + (- \cos t)\hat{j} + (t^2)\hat{k} + C,$$

and

$$\int_{0}^{\pi/2} \vec{r}(t)dt = 2\hat{i} + \hat{j} + \frac{\pi^2}{4}\hat{k}.$$

Recall that all of these concepts are used for particles of motion.

- $\vec{r}(t)$ is the position vector of a moving object,
- $\vec{v} = d\vec{r}/dt$ is the velocity,
- $\|\vec{v}\|$ is the speed,
- $\vec{v}/\|\vec{v}\|$ gives us the direction, and
- $\vec{a} = d\vec{v}/dt = d^2\vec{r}/dt^2$ is the acceleration.

### 10.4 Curvature

Let’s first talk about arc length. **What’s the formula for the length of a straight line?**

$$\sqrt{(x_1-x_0)^2 + (y_1-y_0)^2 + (z_1-z_0)^2} \text{ for } x_0 \leq x \leq x_1.$$ We can approximate the length of a curve by using our straight line formula. To get a better approximation we just use more and more points. This gives us a sum

$$\sum_{n=1}^{N} \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

Then taking the limit of this sum gives us the integral $\int_{a}^{b} \sqrt{dx^2 + dy^2 + dz^2}$, however this integral isn’t in the form that we are used to. We can’t integrate this as is. Lets multiply the integrand by $dt/dt$ to get

$$L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_{a}^{b} \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt = \int_{a}^{b} \|\vec{r}'(t)\| dt.$$ (4)

Ex: Find the length of the arc $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$ from point $(1, 0, 0)$ to $(1, 0, 2\pi)$.

$$\|\vec{r}(t)\| = \sqrt{(- \sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$$

$$\Rightarrow L = \int_{0}^{2\pi} \|\vec{r}'(t)\| dt = \int_{0}^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi.$$