Lecture Five: Poincaré Maps and Structural Stability

Lecture Five Part I: Poincaré Maps. First lets write down a few key definitions.

**Definition 1.** Suppose $\varphi_t(x_*)$ is a periodic orbit of $\dot{x} = f(x)$, where $f$ is continuously differentiable, of period $T$,
\[
\Gamma = \{x \in \mathbb{R}^n : x = \varphi_t(x_*), 0 \leq t \leq T\}.
\] (1)

**Definition 2.** Let $\Sigma$ be the hyperplane orthogonal to $\Gamma$ at $x_*$,
\[
\Sigma = \{x \in \mathbb{R}^n : (x - x_*) \cdot f(x_*) = 0\}.
\] (2)

**Theorem 1.** There is an $\varepsilon > 0$ and a unique function $\tau(x)$, which is continuously differentiable for $x \in B_\varepsilon(x_*)$ such that $\tau(x_*) = T$ and $\varphi_{\tau(x)}(x) \in \Sigma$ for all $x \in B_\varepsilon(x_*)$.

**Definition 3.** For $x \in B_\varepsilon(x_*) \cup \Sigma$, the function $P(x) = \varphi_{\tau(x)}(x)$ is called a Poincaré map for $\Gamma$ at $x_*$. 

Ex Consider the following ODE,
\[
\begin{align*}
\dot{x} &= -y + x(1 - x^2 - y^2) \\
\dot{y} &= x + y(1 - x^2 - y^2)
\end{align*}
\] (3)

We notice that we can write this in polar coordinates via the transformation, $x = r \cos \theta$ and $y = r \sin \theta$,
\[
\begin{align*}
\dot{r} &= r(1 - r^2) \\
\dot{\theta} &= 1
\end{align*}
\] (4)

Now, we notice that the fixed points of the $r$ equation correspond to limit cycles of the $x,y$ equation. Therefore, we have a limit cycle when $r = 1$, which corresponds to $(x,y) = (\cos t, \sin t)^T$. We can solve this via separation,
\[
\begin{align*}
\ln \left(1 + \frac{1}{r(0)^2} - 1\right) e^{-2t}, \\
\theta &= t + \theta(0).
\end{align*}
\]

Let $\theta(0) = \theta_0$, $r(0) = r_0$, and $\Sigma$ be the ray $\theta = \theta_0$ through the origin. Then, $\Sigma$ is orthogonal to $\Gamma$, which has a period of $T = 2\pi$, so the Poincaré map is
\[
r_{n+1} = P(r_n) = \sqrt{1 + \left(\frac{1}{r_n^2} - 1\right)} e^{-4\pi}.
\] (5)
Notice that $r_* = 1$. We wish to use the Poincaré map to find the stability of this limit cycle, so we take the derivative,

$$P'(r_n) = e^{-4\pi r_n^{-3}} \left[ 1 + \left( \frac{1}{r_n^2} - 1 \right) e^{-4\pi} \right]^{-3/2} \Rightarrow |P'(1)| = e^{-4\pi} < 1.$$

In general it’s impossible to write down the Poincaré map explicitly, but there are theorems to help our analysis.

**Lecture Five Part II: Structural Stability.** In class I basically outlined the ideas from my paper. You can access it here: [http://arxiv.org/abs/1306.0436](http://arxiv.org/abs/1306.0436)

Instead of rewriting it here, I will only go through the details of some of the examples I provided in the paper.

**Ex (Homeomorphism)** Consider the function $h : (0, \infty) \to (0, 1)$ defined by $h = (1 + x^2)^{-1}$. Let’s prove that this is a homeomorphism.

**Proof.** First let’s show the inverse exists, i.e. the inverse is well defined on the codomain. We compute the inverse to be $h^{-1} = \sqrt{1 - 1/x}$, which is defined for all $x \in (0, 1)$.

Next let’s show that it is injective (i.e. one-to-one). Suppose $h(a) = h(b)$, then $(1 + a^2)^{-1} = (1 + b^2)^{-1} \Rightarrow 1 + b^2 = 1 + a^2 \Rightarrow b^2 = a^2$, and since $a, b \in (0, \infty)$, $b = a$.

Now let’s show that it is surjective (i.e. onto). Consider $y \in (0, 1)$, then if $h(x) = y$, $x = \sqrt{1 - 1/y} \in (0, \infty)$.

Finally, we show that it is continuous. We see that $h(x = c) = (1 + c^2)^{-1} \in (0, 1)$, and $\lim_{x \to c} h(x) = h(c) \in (0, 1)$.

Similarly the inverse is injective, surjective, and continuous. \hfill \Box

**Ex (Topological Equivalence)** Consider the dynamical systems $\dot{\theta} = \sin \theta$ and $\dot{\varphi} = \cos \varphi$. Let’s prove that these are topologically equivalent on $S^1$.

**Proof.** Notice that if $\varphi = \theta - \pi/2$, our functions are equivalent. So, our homeomorphism is, $h : S^1 \to S^1$ defined by $h(\theta) = \theta - \pi/2$. \hfill \Box