Supplementary problems: 13.3 # 1,3,5,8
Quiz: 13.3

Compulsory problems:
Consider the heat equation
\[ \frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \]  
with the following initial and boundary conditions

(1) [9 pts.] \( u(a, t) = 5 \), \( u_x(b, t) = 0 \). Which of the following initial conditions will yield no solution? Provide reasoning.
(Hint1: Don’t solve the heat equation). (Hint2: If the initial temperature profile (initial condition) does not match the boundary conditions or if there is a jump/essential discontinuity at any point in the initial temperature profile, we can’t solve the heat equation as is.)
(a) \( u(x, 0) = 10 \)
(b) \( u(x, 0) = 5 \)
(c) \( u(x, 0) = \frac{5(x-a)}{a-b} + 5 \)
(d) \( u(x, 0) = \begin{cases} 6 - \frac{x}{a} & \text{for } a < x < (b-a)/2 \\ 6 - \frac{b-a}{2a} & \text{for } (b-a)/2 < x < b \end{cases} \)
(e) \( u(x, 0) = \begin{cases} 5 & \text{for } a < x < (b-a)/2 \\ 4 & \text{for } (b-a)/2 < x < b \end{cases} \)

Hint3: Here are the three possible reasons that you can give for no solution, otherwise the heat equation will have a solution. Occasionally multiple reasons work for the same problem, but you only need to pick one.
- The initial condition doesn’t match the boundary condition at \( x = a \).
- The initial condition doesn’t match the boundary condition at \( x = b \).
- The initial condition has a jump discontinuity in the middle.

(2) [11 pts.] \( u(1, t) = u(\infty, t) = 0; u(x, 0) = 1 \).

Do a change of variables \( x \mapsto \xi \) to put the equation onto a finite domain; i.e. \( u(\xi = 0, t) = u(\xi = 1, t) \). And write down the heat equation with this change of variables; i.e. in terms of \( \xi \).

Hint: Recall if \( \xi = f(x) \)
\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} = f'(x) \frac{\partial u}{\partial \xi} \]  
(2)

(3) [40 pts.]
\[ u(0, t) = u_x(2, t) = 0; \quad u(x, 0) = \begin{cases} x & \text{for } 0 < x < 1 \\ 1 & \text{for } 1 < x < 2 \end{cases} \]  
(3)

Solve the heat equation and write down the complete solution. You can skip the nonessential steps, but please show the integration.

Your homework raw score is: \( \frac{n}{2m} \cdot M + \left( 1 - \frac{n}{2m} \right) \cdot N = N + \frac{n}{2m} (M - N) \). For this homework, \( M = 60 \), \( m = 4 \), \( N \) is the number of compulsory problems you get correct, and \( n \) is the number of supplementary problems you complete. It should be noted that for the supplementary problems I will be looking for full completion, but I won’t take off points for mistakes.