12.2: Fourier Series

This definition allows us to construct a space of functions out of two simple functions. Now equipped with our new machinery we can derive a series representation that is ideal for periodic functions. We did this in class, but here I shall just remind you of the formulas:

**Fourier Series.**

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right];
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) dx
\]

Now lets do some problems. While a lot of these want plotting, we did them in class, so I won’t show them here, but make sure you know how to plot these things.

Ex: Find the Fourier Series of the function

\[
f(x) = \begin{cases} 
1 & -L < x < 0, \\
0 & 0 \leq x < L;
\end{cases}
\]

(a) Sketch it!
(b) We first do \( a_0 \)

\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx = \frac{1}{L} \int_{-L}^{0} dx = 1.
\]

Notice that we always do \( a_0 \) separately. Then we do \( a_n \)

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) dx = \frac{1}{L} \int_{-L}^{0} \cos \left( \frac{n\pi x}{L} \right) dx = \frac{1}{n\pi} \sin \left( \frac{n\pi x}{L} \right) \bigg|_{-L}^{0}
\]

Finally, for \( b_n \)

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) dx = \frac{1}{L} \int_{-L}^{0} \sin \left( \frac{n\pi x}{L} \right) dx = -\frac{1}{n\pi} \cos \left( \frac{n\pi x}{L} \right) \bigg|_{-L}^{0}
\]

\[
= -\frac{1}{n\pi} + \frac{1}{n\pi} \cos(n\pi) = \begin{cases} 
-1 + (-1)^n & n \text{ odd, i.e. } n = 2k + 1; \ k = 0, \pm 1, \pm 2, \ldots \\
-2 & n = 2k; \ k = 0, \pm 1, \pm 2, \ldots
\end{cases}
\]

Then our Fourier series becomes

\[
f(x) = \frac{1}{2} - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{2k+1} \sin \left( \frac{1}{L} (2k+1) \pi x \right).
\]

Ex: Find the Fourier Series of the function \( f(x) = x^2 / 2 \) on \([-2, 2]\)

(a) Plot it!
(b) Again, we do \( a_0 \) first

\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx = \frac{1}{2} \int_{-2}^{2} x^2 dx = \frac{x^3}{12} \bigg|_{-2}^{2} = \frac{4}{3}.
\]
Now to do $a_n$ we need to do by parts twice, which you can do yourselves. I’ll just give the final form of the antiderivative.

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n \pi x}{L} \right) \, dx = \frac{1}{2} \int_{-2}^{2} \frac{x^2}{2} \cos \left( \frac{n \pi x}{2} \right) \, dx = \frac{1}{2} \int_{0}^{2} \frac{x^2}{2} \cos \left( \frac{n \pi x}{2} \right) \, dx$$

$$= \left[ \frac{2x^2}{n \pi} \sin \left( \frac{n \pi x}{2} \right) + \frac{8x}{(n \pi)^2} \cos \left( \frac{n \pi x}{2} \right) - \frac{16}{(n \pi)^3} \sin \left( \frac{n \pi x}{2} \right) \right]^2_{0} = \frac{8}{(n \pi)^2} \cos(n \pi) = (-1)^n \frac{8}{(n \pi)^2}.$$

For $b_n$ we get

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) \, dx = \frac{1}{2} \int_{-2}^{2} \frac{x^2}{2} \sin \left( \frac{n \pi x}{2} \right) \, dx = 0.$$

because we are integrating an odd function on a symmetric interval. Then our Fourier series is

$$f(x) = \frac{2}{3} + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \left( \frac{n \pi x}{2} \right).$$

15) This is a book problem.

First we find $a_0$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \, dx = \frac{1}{\pi} e^x \bigg|_{-\pi}^{\pi} = \frac{1}{\pi} (e^{\pi} - e^{-\pi}) = \frac{2}{\pi} \sinh \pi$$

Then we find $a_n$ via “by parts” using $u = \cos n x \Rightarrow du = -n \sin n x \, dx$ and $dv = e^x \, dx \Rightarrow v = e^x$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos n x \, dx = \frac{1}{\pi} \left[ e^x \cos n x \bigg|_{-\pi}^{\pi} + n \int_{-\pi}^{\pi} e^x \sin n x \, dx \right]$$

Then we do another by parts: $u = \sin n x \Rightarrow du = n \cos n x \, dx$ and $dv = e^x \, dx \Rightarrow v = e^x$

$$\frac{1}{\pi} \left\{ e^x \cos n x \bigg|_{-\pi}^{\pi} + n \left[ e^x \sin n x \bigg|_{-\pi}^{\pi} - n \int_{-\pi}^{\pi} e^x \cos n x \, dx \right] \right\}$$

$$= \frac{1}{\pi} \left\{ (e^\pi - e^{-\pi}) (-1)^n - n^2 \int_{-\pi}^{\pi} e^x \cos n x \, dx \right\} = (-1)^n \frac{2}{\pi} \sinh \pi - \frac{n^2}{\pi} \int_{-\pi}^{\pi} e^x \cos n x \, dx$$

Now we notice that we have $\int_{-\pi}^{\pi} e^x \cos n x \, dx$ on both the right and left hand sides, so we can combine them,

$$\frac{n^2 + 1}{\pi} \int_{-\pi}^{\pi} e^x \cos n x \, dx = (-1)^n \frac{2}{\pi} \sinh \pi \Rightarrow a_n = \frac{(-1)^n}{n^2 + 1} \cdot \frac{2}{\pi} \sinh \pi$$

For $b_n$ we have something similar so I will skip a bunch of steps,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin n x \, dx = \frac{1}{\pi} \left\{ -e^x \sin n x \bigg|_{-\pi}^{\pi} - n \left[ e^x \cos n x \bigg|_{-\pi}^{\pi} + n \int_{-\pi}^{\pi} e^x \sin n x \, dx \right] \right\}$$

$$\Rightarrow \frac{n^2 + 1}{\pi} \int_{-\pi}^{\pi} e^x \sin n x \, dx = (-1)^n \frac{2n}{\pi} \sinh \pi \Rightarrow b_n = \frac{(-1)^n}{n^2 + 1} \cdot \frac{2n}{\pi} \sinh \pi$$

Then the Fourier Series is

$$f(x) = \frac{2}{\pi} \sinh \pi \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} (\cos n x - n \sin n x) \right].$$
12.3: Even and Odd Functions

As we saw for the last problem in the preceding section, it can be useful to know whether or not a function is odd or even. Also, many times we will want the Fourier series of a non-periodic function. In order to do this we need to create a periodic function that includes our non-periodic function. Instead of creating something that is neither odd nor even if we create an even or odd function we can save a lot of time. Before we see these techniques let’s define some terms and develop the theory.

**Definition 1.** Consider the function \( f(x) \) such that \( f(-x) = f(x) \), then \( f \) is said to be even.

**Definition 2.** Consider a function \( f(x) \) such that \( f(-x) = -f(x) \), then \( f \) is said to be odd.

There are some important properties that we should keep in mind.

**Properties.**

- Sum/difference of two even functions is even.
- Sum/difference of two odd functions is odd.
- Sum/difference of an even and an odd function is neither even nor odd.
- Product/quotient of two even functions is even.
- Product/quotient of two odd functions is even.
- Product/quotient of an even function and an odd function is odd.
- If \( f \) is even, \( \int_{-L}^{L} f(x) dx = 2 \int_{0}^{L} f(x) dx \).
- If \( f \) is odd, \( \int_{-L}^{L} f(x) dx = 0 \).

Now we can think of a Fourier cosine series and Fourier sine series. These can be derived straight from the Fourier series equations so it’s best not to memorize these formulas.

**Fourier cosine series.** If \( f \) is an even periodic function generated on \( -L \leq x \leq L \), then \( b_n = 0 \), so

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n \pi x}{L} \right)
\]

\[
a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \left( \frac{n \pi x}{L} \right) dx
\]

**Fourier sine series.** If \( f \) is an odd periodic function generated on \( -L \leq x \leq L \), then \( a_n = 0 \), so

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n \pi x}{L} \right)
\]

\[
b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) dx
\]

For the next few problems we just apply the definition of odd and even functions.

1) Odd
5) Even
6) Neither
**Periodic Extensions.** Suppose a function $f$ is defined only on $[0, L]$. If we want to find the Fourier series of this we need to make a periodic function that “includes” $f$. These are called periodic extensions and can either be odd or even.

For these problems we did the sketching in class. Here I will do the problems that requires calculations

Ex: Find the Fourier Sine Series of $f(x) = L - x$ on $[0, L]$.

(a) Notice that for odd extensions our periodic function of period $2L$ becomes

$$g(x) = \begin{cases} -f(-x) & -L < x < 0, \\ f(x) & 0 < x < L; \end{cases}$$

We know that for odd extensions we’ll get a sine series so we only do the sine calculations,

$$b_n = \frac{2}{L} \int_0^L (L-x) \sin \left( \frac{n \pi x}{L} \right) dx = - (L-x) \frac{2}{n \pi} \cos \left( \frac{n \pi x}{L} \right) \bigg|_0^L - \frac{2}{n \pi} \int_0^L \cos \left( \frac{n \pi x}{L} \right) dx = \frac{2L}{n \pi} + \frac{2L}{(n \pi)^2} \sin \left( \frac{n \pi x}{L} \right) \bigg|_0^L$$

Then our Fourier sine series is

$$f(x) = \frac{2L}{\pi} \sum_{n=1}^\infty \frac{1}{n} \sin \left( \frac{n \pi x}{L} \right).$$

(b) Sketch the solution for $L = 4$.

Ex: Find the Fourier Sine and Cosine series of the following function

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1, \\ 0 & \text{for } 1 < x < 2 \end{cases}$$

(a) Sketch the even and odd extensions of the function.

(b) For the cosine series we have

$$a_0 = \frac{2}{L} \int_0^L f(x) dx = \int_0^1 x dx = \frac{1}{2},$$

and

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{n \pi x}{L} \right) dx = \int_0^1 x \cos \left( \frac{n \pi x}{2} \right) dx = \frac{2}{n \pi} \sin \left( \frac{n \pi x}{2} \right) + \frac{4}{(n \pi)^2} \cos \left( \frac{n \pi x}{2} \right) \bigg|_0^1$$

$$= \frac{2}{n \pi} \sin \left( \frac{n \pi}{2} \right) + \frac{4}{(n \pi)^2} \cos \left( \frac{n \pi}{2} \right) - \frac{4}{(n \pi)^2}. $$

Notice that for this problem we can’t simplify the indices in any reasonable manner, so we leave it as is. So the Fourier cosine series is

$$f(x) = \frac{1}{4} + \sum_{n=1}^\infty \left[ \frac{2}{n \pi} \sin \left( \frac{n \pi x}{2} \right) + \frac{4}{(n \pi)^2} \cos \left( \frac{n \pi x}{2} \right) - \frac{4}{(n \pi)^2} \right] \cos \left( \frac{n \pi x}{2} \right).$$

Now, for the sine series we have

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n \pi x}{L} \right) dx = \int_0^1 x \sin \left( \frac{n \pi x}{2} \right) dx = -\frac{2}{n \pi} \cos \left( \frac{n \pi x}{2} \right) + \frac{4}{(n \pi)^2} \sin \left( \frac{n \pi x}{2} \right) \bigg|_0^1$$

$$= -\frac{2}{n \pi} \cos \left( \frac{n \pi}{2} \right) + \frac{4}{(n \pi)^2} \sin \left( \frac{n \pi}{2} \right)$$

Then our Fourier series is

$$f(x) = \sum_{n=1}^\infty \left[ -\frac{2}{n \pi} \cos \left( \frac{n \pi x}{2} \right) + \frac{4}{(n \pi)^2} \sin \left( \frac{n \pi x}{2} \right) \right] \sin \left( \frac{n \pi x}{2} \right).$$