13.3 Heat Equation Examples

Consider the heat equation with a generic initial condition,
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}; \quad u(x, 0) = f(x). \] (1)

with the following boundary conditions

Ex: \( u(0, t) = u(L, t) = 0. \)

Solution: We make the Ansatz, \( u(x, t) = T(t)X(x) \). Then we plug this into our heat equation
\[ u_t = T'(t)X(x), \quad u_{xx} = T(t)X''(x) \Rightarrow T'X = kTX'' \Rightarrow \frac{T'}{kT} = \frac{X''}{X}. \]

Since the LHS is a function of \( t \) alone, and the RHS is a function of \( x \) alone, and since they are equal, they must equal a constant. Let's call it \(-\lambda^2\). Then we have
\[ \frac{T'}{kT} = \frac{X''}{X} = -\lambda^2. \] (2)

Notice that I call this from the get go because in our Sturm-Liouville problems the negative eigenvalue case always gave us trivial solutions. Here we bypass that by automatically assuming a positive eigenvalue \( \lambda^2 \). Now we must solve the two differential equations.

The \( T \) equation is the easiest to solve
\[ \frac{T'}{kT} = -\lambda^2 \Rightarrow T' = -k\lambda^2 T \Rightarrow \int \frac{dT}{T} = -k\lambda^2 \int dt \Rightarrow \ln T = -k\lambda^2 t \Rightarrow T = e^{-k\lambda^2 t}. \]

Notice that we don't include the constant in front of the exponential, and that is because the \( X \) equation will have constants, and we would simply by multiplying constants to reduce it to one constant anyway, so I choose to leave it out from the beginning. You don't have to though.

Now, we solve the \( X \) equation by recalling our Sturm-Liouville problems
\[ \frac{X''}{X} = -\lambda^2 \Rightarrow X'' + \lambda^2 X = 0 \Rightarrow X = A \cos \lambda x + B \sin \lambda x \text{ for } \lambda \neq 0 \text{ and } X = c_1 x + c_2 \text{ for } \lambda = 0. \]

If we look at the \( \lambda = 0 \) case we have \( X(0) = c_2 = 0 \) and \( X(L) = Lc_1 = 0, \) so \( X \equiv 0. \)

Now we look at the \( \lambda \neq 0 \) case. \( X(0) = A = 0 \) and
\[ X(L) = X(L) = B \sin \lambda x = 0 \Rightarrow \lambda = \frac{n\pi}{L} \Rightarrow X_n = B_n \sin \frac{n\pi x}{L} \text{ and } T_n = e^{-k(n\pi/L)^2 t}. \]

Next we combine the \( T \) and \( X \) solutions to get the general solutions,
\[ u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}. \] (3)

And we can solve for the constants using the principles from Fourier series with the initial condition. Since this is a Fourier sine series we have
\[ u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x) \Rightarrow B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \]

Then our full solution is
\[ u(x, t) = \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t} \int_0^L f(x) \sin \frac{n\pi x}{L} dx. \] (4)
Ex: \( u_x(0, t) = u_x(L, t) = 0 \).

**Solution:** We know from the first example that \( T = e^{-k\lambda^2 t} \).

For the \( X \) equation we need to look at our two cases. For \( \lambda = 0 \) we have \( X = c_1 x + c_2 \), and \( X'(x) = c_1 \), so for both boundaries \( X'(0) = c_1 = X'(L) \). These leaves us with a constant \( X = c_2 \).

For the \( \lambda \neq 0 \) case we have \( X = A \cos \lambda x + B \sin \lambda x \Rightarrow X' = -\lambda A \sin \lambda x + \lambda B \cos \lambda x \)

Then we get \( X'(0) = \lambda B = 0 \) and \( X'(L) = -\lambda A \sin \lambda L = 0 \Rightarrow \lambda = n\pi \Rightarrow X_n = A_n \cos \frac{n\pi x}{L} \) and \( T_n = e^{-k(n\pi)^2 t} \)

Next we combine the \( T \) and \( X \) solutions to get our general solution

\[
\begin{align*}
  u(x, t) &= c_2 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-k(n\pi)^2 t} \\
\end{align*}
\]

(5)

Now we find our coefficients by invoking the initial condition and using Fourier Series

\[
\begin{align*}
  u(x, 0) &= c_2 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} = f(x) \\
\end{align*}
\]

This gives us

\[
\begin{align*}
  c_2 &= \frac{1}{L} \int_0^L f(x) dx \\
\end{align*}
\]

and

\[
\begin{align*}
  A_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \\
\end{align*}
\]

Combining everything we get the full solution

\[
\begin{align*}
  u(x, t) &= \frac{1}{L} \int_0^L f(x) dx + 2 \sum_{n=1}^{\infty} \cos \frac{n\pi x}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \\
\end{align*}
\]

(6)

Ex: Now lets think of heat transfer in a circle. If we go around in one direction we hit \( x = -L \) and in the other direction \( x = L \), but these are the same point. So we get the following boundary conditions

\[
\begin{align*}
  u(-L, t) = u(L, t), u_x(-L, t) = u_x(L, t) \\
\end{align*}
\]

(7)

**Solution:** We know from the previous two problems that our solutions will be

\[
\begin{align*}
  T &= e^{-k\lambda^2 t} \\
  X &= c_1 x + c_2 \text{ for } \lambda = 0 \\
  X &= A \cos \lambda x + B \sin \lambda x \text{ for } \lambda \neq 0 \\
\end{align*}
\]

For \( \lambda = 0 \), \( X(L) = c_1 L + c_2 \) and \( X(-L) = -c_1 L + c_2 \), so \( c_1 = 0 \). And the derivative is trivially satisfied.
For $\lambda \neq 0$,

$$X(L) = X(-L) \Rightarrow A \cos \lambda L + B \sin \lambda L = A \cos \lambda L - B \sin \lambda L \Rightarrow \sin \lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L}$$

And

$$X'(L) = X'(-L) \Rightarrow -\lambda A \sin \lambda L + \lambda B \sin \lambda L = \lambda A \sin \lambda L + \lambda B \cos \lambda L \Rightarrow \sin \lambda L = 0$$

But we already showed this. So, we need to keep both coefficients. Then our solution for $X$, which as we saw in previous conditions (for the heat equation) is just the initial condition of the general solution, is

$$X = c_2 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} = u(x, 0) = f(x) \quad (8)$$

Now we use Fourier series to solve for the coefficients,

$$c_2 = \frac{1}{L} \int_{0}^{L} f(x) dx$$

$$A_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

$$B_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

Putting everything back into the general solution gives us

$$u(x, t) = \frac{1}{L} \int_{0}^{L} f(x) dx + \frac{2}{L} \sum_{n=1}^{\infty} \cos \frac{n\pi x}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} dx + \sin \frac{n\pi x}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx \quad (9)$$