4.1 - 4.3 Vector spaces and subspaces

Here we saw some definitions and how they apply to some examples.

**Definition 1.** The vectors $v_1, \ldots, v_n$ are said to be Linearly Independent if $c_1 v_1 + \cdots + c_n v_n \neq 0$ when $c_i \neq 0$ for $i = 1, \ldots, n$; otherwise they are said to be Linearly Dependent.

**Definition 2.** The expression $c_1 v_1 + \cdots + c_n v_n$ is said to be a Linear Combination of $v_1, \ldots, v_n$.

A vector space is simply a space that contains all of the axioms of vector addition and scalar multiplication, and is self-contained; i.e., addition and scalar multiplication of any combination of vectors will produce a vector in that space.

**Definition 3.** A subspace of a vector space is a nonempty subset that satisfies the requirements for a vector space: Linear combinations stay in the subset;
(i) if we add any vectors $x$ and $y$ in the subspace $x + y$ is in the subspace,
(ii) if we multiply any vector $x$ in the subspace by any scalar $c$, $cx$ is in the subspace.

Now let’s do some problems from pg. 173.

1) $W$ is clearly nonempty and a subset of $V$. We just have to check the properties listed in Def. 3.
   (i)
   \[
   \begin{pmatrix}
   x_1 \\
   x_2 \\
   x_3 \\
   0
   \end{pmatrix} + \begin{pmatrix}
   y_1 \\
   y_2 \\
   y_3 \\
   0
   \end{pmatrix} = \begin{pmatrix}
   x_1 + y_1 \\
   x_2 + y_2 \\
   x_3 + y_3 \\
   0
   \end{pmatrix}
   \]
   By the axioms of arithmetic $x_i + y_i$ will be real numbers, and the last entry is zero, so this is in $W$.
   (ii)
   \[
   c \begin{pmatrix}
   x_1 \\
   x_2 \\
   x_3 \\
   0
   \end{pmatrix} = \begin{pmatrix}
   cx_1 \\
   cx_2 \\
   cx_3 \\
   0
   \end{pmatrix}
   \]
   Again, by the axioms of arithmetic $cx_i$ will be real numbers, and the last entry is zero, so this is in $W$ as well.
   Since both properties are satisfied, $W$ is a subspace of $V$.

2) Just as the previous problem
   (i)
   \[
   \begin{pmatrix}
   x_1 \\
   x_2 \\
   4x_1 - 5y_1
   \end{pmatrix} + \begin{pmatrix}
   y_1 \\
   y_2 \\
   y_3
   \end{pmatrix} = \begin{pmatrix}
   x_1 + y_1 \\
   x_2 + y_2 \\
   4(x_1 + x_2) - 5(y_1 + y_2)
   \end{pmatrix}
   \]
   By the axioms of arithmetic all three entries will be real, thus matching the definition of the set $W$, and hence the vector is in $W$.
   (ii)
   \[
   c \begin{pmatrix}
   x_1 \\
   x_2 \\
   4x_1 - 5y_1
   \end{pmatrix} = \begin{pmatrix}
   cx_1 \\
   cx_2 \\
   4cx_1 - 5cy_1
   \end{pmatrix}
   \]

7) Here both properties can be violated. For property (ii),
   \[
   c \begin{pmatrix}
   x \\
   y \\
   -1
   \end{pmatrix} = \begin{pmatrix}
   cx \\
   cy \\
   -c
   \end{pmatrix}
   \]
   In general $-c \neq -1$, so this vector cannot be in $W$. Hence, $W$ is not a subspace of $V$.

9) Here only property (ii) is violated,
   \[
   \sqrt{2} \begin{pmatrix} 1 \\
   1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\
   \sqrt{2} \end{pmatrix} \notin Q,
   \]
   therefore is not in $W$, and $W$ is not a subspace of $V$. 

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