6.5 Impulse Functions

An impulse is a change of momentum over a period of time. This can range from hitting a baseball to a punch to the face. The following plot gives an illustration of this,

\[ p = \int_{-\tau}^{\tau} g(t) dt = \int_{-\tau}^{\tau} 1/(2\tau) dt = 1. \]

Notice that we can make \( \tau \) smaller and keep the momentum at \( p = 1 \) such as in the following plot,
In fact,
\[
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} g(t) dt = 1.
\]
Notice this is 0 everywhere except at \( t = 0 \). Now, if we can do this at \( t = 0 \) we can define a “function” with this property for any \( t = t_0 \),
\[
\int_{-\infty}^{\infty} \delta(t - t_0) dt = 1; \quad \delta(t - t_0) = 0 \quad \forall t \neq t_0
\]
called the Dirac delta function, however this isn’t a function, but rather a distribution. Doing this for \( t_0 > 0 \) will allow us to employ laplace transforms. Notice that we can write the delta function as the following limit,
\[
\delta(t - t_0) = \lim_{\epsilon \to 0} \begin{cases} 
0 & t \leq t_0 - \epsilon, \\
\frac{1}{2\epsilon} (t - t_0 - \epsilon < t < t_0 + \epsilon), \\
0 & t \geq t_0 + \epsilon;
\end{cases} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} (u_{t_0-\epsilon}(t) - u_{t_0+\epsilon}(t)).
\]
Now we take the laplace transform,
\[
\mathcal{L}\{\delta(t - t_0)\} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \cdot \frac{1}{s} \left( e^{-(t_0+\epsilon)s} - e^{-(t_0-\epsilon)s} \right) = e^{-t_0s} \lim_{\epsilon \to 0} \frac{1}{\epsilon s} \cdot \frac{1}{2} \left( e^{\epsilon s} - e^{-\epsilon s} \right)
\]
\[
= e^{-t_0s} \frac{1}{\epsilon} \lim_{\epsilon \to 0} \frac{\sinh \epsilon s}{\epsilon s} = e^{-t_0s}.
\]
Now, let’s do some problems.
4) We take the laplace transform of the entire ODE (Plot on left),
\[
-y'(0) - sy(0) + s^2Y - Y = -20e^{-3s} \Rightarrow (s^2 - 1)Y = -20e^{-3s} \Rightarrow Y = \frac{1}{s^2 - 1} (-20e^{-3s} + s) \\
\Rightarrow y = \cosh t - 20 \sinh(t - 3)u_3(t).
\]
8) We take the laplace transform of the entire ODE (Plot on right),
\[
-y'(0) - sy(0) + s^2Y + 4Y = 2e^{-(\pi/4)s} \Rightarrow Y = \frac{2}{s^2 + 4} e^{-(\pi/4)s} \\
\Rightarrow y = \sin(2(t - \pi/4))u_{\pi/4}(t) = (\cos 2t)u_{\pi/4}(t).
\]
11) As per usual, 

\[(s^2 + 2s + 2)Y = \frac{s}{s^2 + 1} + e^{-\pi s} \Rightarrow Y = \frac{s}{(s^2 + 1)(s^2 + 2s + 2)} + \frac{e^{-\pi s}}{s^2 + 2s + 2}\]

We employ partial fractions, 

\[\frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 2s + 2} = \frac{s}{(s^2 + 1)(s^2 + 2s + 2)} \Rightarrow \frac{A}{s^2 + 1} + \frac{2A}{s^2 + 2s + 2} + \frac{B}{s^2 + 1} + \frac{2B}{s^2 + 2s + 2} + \frac{Cs + D}{s^2 + 2s + 2} + D = s \]

\[\Rightarrow (A + C)s^3 + (2A + B + D)s^2 + (2A + 2B + C)s + (2B + D) = s.\]

From this we get

\[A = 1/5 = -C,\ B = 2/5,\ \text{and}\ D = -4/5,\ \text{so}\]

\[Y = \frac{1}{5} \left[ \frac{s}{s^2 + 1} + \frac{2}{s^2 + 1} - \frac{s + 4}{s^2 + 2s + 2} \right] + e^{-\pi s} \frac{1}{(s + 1)^2 + 1}.\]

Furthermore,

\[\frac{s + 4}{(s + 1)^2 + 1} = \frac{s + 1}{(s + 1)^2 + 1} + \frac{3}{(s + 1)^2 + 1}.\]

Then,

\[y = \frac{1}{5} \cos t + \frac{2}{5} \sin t - \frac{1}{5} e^{-t}(\cos t + 3 \sin t) + e^{-t\pi/2} \sin(t - \pi/2)u_{\pi/2}(t).\]

### 6.6 Convolutions

To derive this we need knowledge of Calc III, which I know not everyone had, so we will just define it. A convolution is the following operator,

\[(f * g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau\]

The laplace transform is as follows,

\[\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} \tag{4}\]

It should be noted that this is similar to multiplication and has some of the same properties:

1) \(f * g = g * f\)  
2) \(f * (g_1 + g_2) = f * g_1 + f * g_2\)  
3) \((f * g) * h = f * (g * h)\).

Now, let's do some problems,

7) We take the laplace transform of sine and cosine and then multiply them together,

\[\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}, \mathcal{L}\{\cos t\} = \frac{s}{s^2 + 1} \Rightarrow \mathcal{L}\{f(t)\} = \frac{s}{(s^2 + 1)^2}\]

11) Here we take the inverse. We know the transform of sine from above and the inverse transform of \(G(s)\). So we get,

\[\mathcal{L}^{-1}\{F(s)\} = \int_0^t \sin t g(t - \tau)d\tau.\]
Here we take the laplace transform of the entire ODE,

\[-y'(0)-sy(0)+s^2Y-4y(0)+4sY+4Y = G(s) \Rightarrow (s^2+4s+4)Y = 2s+5+G(s) \Rightarrow Y = \frac{2s+5}{(s+2)^2} + \frac{G(s)}{(s+2)^2}.\]

We employ partial fractions,

\[\frac{A}{s+2} + \frac{B}{(s+2)^2} = \frac{2s+5}{(s+2)^2} \Rightarrow As + 2A = 2s + 5.\]

This gives, \(A = 2, B = 1.\) Then we get,

\[Y = \frac{2}{s+2} + \frac{1}{(s+2)^2} + \frac{G(s)}{(s+2)^2}.\]

Taking the inverse transform gives,

\[y = 2e^{-2t} + te^{-2t} + \int_0^t \tau e^{-2\tau} g(t-\tau) d\tau.\]

Again,

\[-y''(0) - sy'(0) + s^2Y - y''(0) + sY + \frac{5}{4}Y = \frac{1}{s} - \frac{1}{s} e^{-\pi s} \Rightarrow (s^2 + s + 5/4)Y = s + \frac{1}{s} - \frac{1}{s} e^{-\pi s}\]

\[\Rightarrow Y = \frac{s}{s^2 + s + 5/4} + \frac{1 - e^{-\pi s}}{(s^2 + s + 5/4)} = \frac{s + 1/2}{(s + 1/2)^2 + 1} - \frac{1/2}{(s + 1/2)^2 + 1} + \frac{1}{s + 1/2} + \frac{1}{s} e^{-\pi s}\]

\[\Rightarrow y = -t^{-1/2} \cos t - \frac{1}{2} e^{-t/2} \sin t + \int_0^t e^{-\tau/2} \sin \tau (1 - u_\tau(t-\tau)) d\tau.\]

I made a small error in class which changed the problem, so make sure you go over this one.

### 7.1 Introduction to systems of first order ODEs

In class we went through the example of a simple pendulum. I went redo that example here, but what we take out of that is the simple pendulum is governed by the ODE: \(\theta'' + (g/L) \sin \theta = 0.\) And we can convert this into a system of two first order ODEs by letting \(\omega = \theta',\) then \(\theta' = \omega\) and \(\omega' = -(g/L) \sin \theta.\) By doing this we could extract a lot of necessary information to an otherwise unsolvable (with the methods we know) problem. We can use this trick for other problems as done below,

1. Let \(v = u',\) then \(v'' = -v/2 + 2u.\)
2. Let \(v = u',\) then \(v'' = -v/2 + (1/4 - t^2)u/t^2.\)
3. Let \(v = u',\) then \(v'' = g(t) - p(t)v - q(t)u\) and \(u(0) = v(0) = u_0, v(0) = u_0'.\)
4. Notice \(x_2 = (x_1 - x_1')/2,\)

\[(x_1 - x_1')/2' = 3x_1 - 4((x_1 - x_1')/2) \Rightarrow x_1'' = 2x_1 + 4x_1' \Rightarrow x_1'' + 3x_1' + 2x_1 = 0; \ x(0) = -1, \ x_1(0) = -5.\]

Now we solve for \(x_1,\)

\[r^2 + 3r + 2 = (r+2)(r+1) = 0 \Rightarrow r = -2, -1, \text{ then}\]

\[x_1 = c_1 e^{-t} + c_2 e^{-2t}.\]

From the initial conditions we get, \(x_1(0) = c_1 + c_2 = -1 \text{ and } x_1'(0) = -c_1 - 2c_2 = -5,\) then \(c_2 = 6, c_1 = -7.\) Now to solve for \(x_2\) we plug \(x_1\) into the first equation where we have \(x_2\) as a function of \(x_1\) and \(x_1'\) to get,

\[x_1 = 6e^{-t} - 7e^{-2t}; \ x_2 = -7e^{-t} + 9e^{-2t}.\]