(1) \( y = x^r \Rightarrow y' = rx^{r-1} \Rightarrow y'' = r(r-1)x^{r-2}. \) Plugging this into the ODE gives us \( r(r-1)x^r - 2x^r = (r^2 - r - 2)x^r = 0, \) so \( r^2 - r - 2 = 0 \Rightarrow \{r = 2, -1\}, \) then our characteristic solution is \( y_c = c_1x^2 + c_2x^{-1}. \)

(b) From the characteristic solution we compute our Wronskian,

\[
W(y_1, y_2) = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix} = -1 - 2 = -3
\]

Don’t forget to put the equation into standard form, which gives us \( f(x) = 3x^3 + 1. \) Then we plug into our variation of parameters formula to get

\[
y = -x^2 \int \frac{(3x^3 + 1)}{-3} \, dx + x^{-1} \int \frac{(3x^3 + 1)}{-3} \, dx = -x^2 \int \left(-x^{-4} - \frac{1}{3}x^{-1}\right) \, dx + x^{-1} \int \left(-x^{-4} - \frac{1}{3}x^{-1}\right) \, dx
\]

(2) \( y = vy_1 \Rightarrow y' = v'y_1 + vy'_1 \Rightarrow y'' = v''y_1 + 2v'y'_1 + vy''_1. \)

\[
(t-1)y'' - ty' + y = (t-1)\left[v''y_1 + 2v'y'_1 + vy''_1\right] - t\left[v'y_1 + vy'_1\right] + vy_1 = (t-1)\left[e^ty'' + 2e^ty'ight] - te^ty' = 0
\]

\[
\Rightarrow (t-1)e^ty'' + [2e^t(t-1) - te^t]v' = (t-1)e^ty'' + (t-2)e^ty' = 0 \Rightarrow v'' = \frac{t-2}{t-1}v'.
\]

Let \( u = v', \)

\[
u' = \left[\frac{1}{t-1} - 1\right]u \Rightarrow \int \frac{du}{u} = \int \left[\frac{1}{t-1} - 1\right] \, dt \Rightarrow \ln|u| = \ln|t-1| - t + C_0 \Rightarrow u = k(t-1)e^{-t} \Rightarrow v = k \int (1-t)e^{-t} \, dt.
\]

By parts: \( u = t-1 \Rightarrow du = dt \) and \( dv = e^{-t} \Rightarrow v = -e^{-t}, \)

\[
v = k \left[(t-1)e^{-t} + \int e^{-t} \, dt\right] = k \left[(t-1)e^{-t} - e^{-t} + C_1\right] \Rightarrow y = k_1 t + C_2 e^t \Rightarrow y_2 = t.
\]

(b) \( W(y_1, y_2) = C \exp\left(-\int q(s) \, ds\right). \)

\[
-\int q(s) \, ds = \int_t^1 \frac{s}{s-1} \, ds = \int_t^1 \frac{(s-1) + 1}{s-1} \, ds = \int_1^t \frac{1}{s-1} = \ln|t-1| + t \Rightarrow W(y_1, y_2) = C(t-1)e^t.
\]

(c) Put it into standard form:

\[
(t-1)y'' - ty' + y = \ln(t) \tan(t) \Rightarrow y'' - \frac{t}{t-1}y' + \frac{1}{t-1}y = \frac{\ln(t) \tan(t)}{t-1}.
\]

The discontinuities are at \( t = 0, 1, \pm n\pi/2, \) however the initial condition is at \( t = 1/2, \) so the only possible interval is \( t \in [0, 1]. \)

(3) \( r^2 - 10r + 25 = (r - 5)^2 = 0 \Rightarrow r = 5 \Rightarrow y = (c_1 + c_2x)e^{5x}. \) Our guess for the particular solution is \( y_p = Ae^{5x}, \) however we have a repeat with repeated roots, so our particular solution actually is

\[
y_p = Ax^2e^{5x} \Rightarrow y_p' = 5Ax^2e^{5x} + 2Axe^{5x} \Rightarrow y_p'' = 25Ax^2e^{5x} + 20Axe^{5x} + 2Ae^{5x}
\]

\[
\Rightarrow 25Ax^2e^{5x} + 20Axe^{5x} + 2Ae^{5x} - 50Ax^2e^{5x} - 20Axe^{5x} + 25Ax^2e^{5x} = 2Ae^{5x} = 10e^{5x}.
\]

Then, \( A = 5, \) so the particular solution is \( y_p = 5x^2e^{5x}, \) and the general solution is \( y = (c_1 + c_2x + 5x^2)e^{5x}. \) Plugging in the initial conditions gives us \( y(0) = c_1 = 0 \Rightarrow y = (c_2x + 5x^2)e^{5x}. \) For the second initial condition we take the derivative: \( y' = (c_2 + 10x)e^{5x} + 5(c_2x + 5x^2)e^{5x}, \) then \( y'(0) = c_2 = 1, \) so the solution is \( y = (x + 5x^2)e^{5x}. \)
We have repeats with the sines and cosines, and since it didn’t tell us to simplify or solve for anything I will just multiply both blocks of terms by $t$ and leave it as,

$$y_p = t(a_1t + a_0)(B_1 \cos 2t + B_2 \sin 2t) + t(b_1t + b_0)(D_1 \cos 2t + D_2 \sin 2t)$$

(b) $y_c = c_1 + c_2 e^{-2t}$. Our guess for the particular solution is $y_p = a_0 + a_1 e^{-2t}$, so we have a repeat on both terms, then it becomes

$$y_p = a_0t + a_1 t e^{-2t} \Rightarrow y'_p = a_0 + a_1 e^{-2t} - 2a_1 t e^{-2t} \Rightarrow y''_p = -4a_1 e^{-2t} - 4a_1 t e^{-2t}$$

$$\Rightarrow -4a_1 e^{-2t} + 4a_1 t e^{-2t} = 2a_0 + 2a_1 e^{-2t} - 4a_1 t e^{-2t} = 2a_0 - 2a_1 e^{-2t} = 3e^{-2t}$$

Then $a_0 = 3/2$ and $a_1 = -1/2$. So the general solution is $y = c_1 + c_2 e^{-2t} + \frac{3}{2} t - \frac{1}{2} t e^{-2t}$. The first initial condition gives us $y(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$. Plugging it back in gives $y = c_1 - c_1 e^{-2t} + \frac{3}{2} t - \frac{1}{2} t e^{-2t}$. Plugging in the second initial condition gives $y'(0) = 2c_1 + 1 = 0 \Rightarrow c_1 = -1/2$ and we get $y = -\frac{1}{2} + \frac{1}{2} e^{-2t} + \frac{3}{2} t - \frac{1}{2} t e^{-2t}$.

(5) $m = 1/2, k = 16/(8/17) = 34, \gamma = 2$, then our IVP is

$$\frac{1}{2} u'' + 2u' + 34u = 0; \ u(0) = 0, \ u'(0) = -\frac{1}{2}$$

We have $r = -2 \pm \sqrt{4 - 68} = -2 \pm 8i \Rightarrow u = e^{-2t}[A \cos 8t + B \sin 8t]$. Plugging in the first initial condition gives $u(0) = A = 0$. The derivative is $u' = -2e^{-2t}B \sin 8t + 8Be^{-2t} \cos 8t$. The second initial condition gives us $u'(0) = 8B = -1/2 \Rightarrow B = -1/16$. Our solution is $u = -\frac{1}{16} e^{-2t} \sin 8t$.

(6) $m = 1/8, k = 4/((3/2)/12) = 32$, then our IVP is

$$\frac{1}{8} u'' + 32u = 3 \cos 15t; \ u(0) = u'(0) = 0$$

Then we get $(1/8)r^2 + 32 = 0 \Rightarrow r^2 = -2^2 \cdot 2 \cdot 4^2 \Rightarrow r = \pm 16i$. So, the characteristic solution is $u_c = A_1 \cos 16t + A_2 \sin 16t$. We will see that there won’t be any repeats, so our particular solution is

$$u_p = B_1 \cos 15t + B_2 \sin 15t \Rightarrow u''_p = -15^2 B_1 \cos 15t - 15^2 B_2 \sin 15t$$

$$\Rightarrow -\frac{15^2}{8} B_1 \cos 15t - \frac{15^2}{8} B_2 \sin 15t + 32B_1 \cos 15t + 32B_2 \sin 15t$$

$$= \left(32 - \frac{15^2}{8}\right) B_1 \cos 15t + \left(32 - \frac{15^2}{8}\right) B_2 \sin 15t = 3 \cos 15t.$$ 

Then $B_2 = 0$ and $B_1 = 3/(16^2 - 15^2) = 3/31 \Rightarrow u_p = (3/31) \cos 15t$. So the general solution is $u = A_1 \cos 16t + A_2 \sin 16t + (3/31) \cos 15t$. The initial conditions give us $u(0) = A_1 + 3/31 = 0 \Rightarrow A_1 = -3/31$ and $u'(0) = 16A_2 = 0$, then our solution is $u = \frac{3}{31} [\cos 15t - \cos 16t]$. However in order to plot it we need to use our trig identities with the definitions: $a = 31t/2$ and $b = t/2$, then

$$\cos 15t = \cos(a - b) = \cos a \cos b + \sin a \sin b; \ \cos 16t = \cos(a + b) = \cos a \cos b - \sin a \sin b$$

$$\Rightarrow \cos 15t - \cos 16t = 2 \sin a \sin b = 2 \sin \frac{31}{2} t \sin \frac{1}{2} t$$

Now our solution is in a form that can be plotted: $u = \frac{6}{31} \sin \frac{32}{2} t \sin \frac{1}{2} t$. 

(4) (a) $y_c = A_1 \cos 2t + A_2 \sin 2t$. Our guess at the particular solution is

$$y_p = (a_2t^2 + a_1t + a_0)(B_1 \cos 2t + B_2 \sin 2t) + (b_1t + b_0)(D_1 \cos 2t + D_2 \sin 2t)$$

We have repeats with the sines and cosines, and since it didn’t tell us to simplify or solve for anything I will just multiply both blocks of terms by $t$ and leave it as,