Math 4350 Rahman
Exam I Review

Most important things to know: induction, proving limits, using limit laws.

**Definition 1.** A sequence \( \{x_n\} \subseteq \mathbb{R} \) converges if there is an \( x \in \mathbb{R} \) such that
For every \( \varepsilon > 0 \), there is an \( N \) such that \( |x - x_n| \leq \varepsilon \) for all \( n \geq N \);
otherwise it diverges. We call this \( x \) the limit of \( \{x_n\} \).

**Theorem 1.** If \( x_n \to p \), then \( \{x_n\} \) is bounded.

**Theorem 2** (Squeeze). For \( \{x_n\}, \{y_n\}, \{z_n\} \subseteq \mathbb{R} \), if \( x_n \leq y_n \leq z_n \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} z_n \), then \( \{y_n\} \) is convergent and \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n \).

**Theorem 3.** If \( x_n \to p \), \( |x_n| \to |p| \).

**Theorem 4** (Monotone Convergence). A monotone sequence converges if and only if it is bounded, and if \( \{x_n\} \) is bounded and increasing,

\[
\lim_{n \to \infty} x_n = \sup \{x_n : n \in \mathbb{N}\},
\]

and if \( \{y_n\} \) is bounded and decreasing,

\[
\lim_{n \to \infty} y_n = \inf \{y_n : n \in \mathbb{N}\}.
\]

**Property 1** (Divergence criteria). If \( \{x_n\} \subseteq \mathbb{R} \), then it diverges if
(1) it is unbounded, or
(2) it has convergent subsequences with differing limits.

**Lemma 1** (Monotone subsequences). If \( \{y_n\} \subseteq \mathbb{R} \), it has a monotone subsequence.

**Theorem 5** (Bolzano–Weierstrass). If \( \{x_n\} \subseteq \mathbb{R} \) is bounded, it has a convergent subsequence.

**Lemma 2.** If \( \{x_n\} \subseteq \mathbb{R} \) converges, then for all \( \varepsilon > 0 \), there is an \( N \in \mathbb{N} \) such that \( |x_m - x_n| < \varepsilon \) for all \( n, m \geq N \).

**Definition 2.** A sequence \( \{x_n\} \subseteq \mathbb{R} \) is called a Cauchy sequence if
for all \( \varepsilon > 0 \), there is an \( N \in \mathbb{N} \) such that \( |x_m - x_n| < \varepsilon \) for all \( n, m \geq N \).
And this property is called the Cauchy criterion.

**Theorem 6** (Cauchy sequences). In \( \mathbb{R} \) every Cauchy sequence converges.

**Induction:** For induction make sure you prove the base case first, then make the inductive assumption, then prove the inductive step. It is not enough to show it for a few cases, nor is it enough to just show the inductive step (until after you graduate that is).

**Proving limits:** To prove the existence of a limit, first do a little scratch work to see what \( N(\varepsilon) \) has to be to make your entire quantity less than \( \varepsilon \), but be careful to make sure that \( N \) is increasing as \( \varepsilon \) is decreasing, otherwise you picked the wrong \( N \).

**Definition 3.** Let \( x_0, \varepsilon \in \mathbb{R} \) such that \( \varepsilon > 0 \). Then the \( \varepsilon \)-neighborhood (ball) around \( x_0 \) is \( B_\varepsilon(x_0) := \{x \in \mathbb{R} : |x - x_0| < \varepsilon\} \).

Notice that in the real line, a neighborhood or ball, is just an open interval, so \( B_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon) \).

Most important homework problems: Compulsory: Hw2 # 1; Hw3 # 3-6; Hw4 All;
Supplementary: 1.2 # 1; 1.2 # 26; 2.2 # 16; 3.1 # 5; 3.2 # 20, 23; 3.3 # 1-3; 3.4 # 3, 9.
Additional problems you can look at: 1.2 # 2 - 5, 7 - 9, 13 - 15, 18; 3.1 # 4, 6, 16, 17; 3.2 # 1, 5, 6, 9 - 17; 3.3 # 4 - 7, 11, 12; 3.4 # 7, 8.

The exam is organized as follows: 45 points will be mainly calculations with the use of a little bit of theory (e.g. proving a limit), 45 points will be using theorems/logic to prove the results, and 10 points will be a completely theoretical problem involving sets, topology, and sequences all in one.

**Note:** I will never ask you to prove a theorem. I don’t want you memorizing proofs!