1. Sets and Functions

You have probably seen a lot of these things in intro to proofs, but it is worth reviewing.

**Definition 1.** Two sets \( A \) and \( B \) are said to be equal (notated as \( A = B \)) if they contain the same elements.

We showed examples of finite sets in class.

To prove \( A = B \), for simple sets we can simply match the elements, however for more complex sets we need to show \( A \subseteq B \) (proper subset) and \( B \subseteq A \). To prove \( A \subseteq B \), we suppose \( x \in A \) and show \( x \in B \).

1.1. Some important sets.

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\begin{align*}
\mathbb{N} &:= \{1, 2, 3, \ldots, n, \ldots\} \\
\mathbb{Z} &:= \{0, \pm 1, \pm 2, \ldots, \pm n, \ldots\} \\
\mathbb{Q} &:= \left\{\frac{n}{m} : m, n \in \mathbb{Z}, n \neq 0\right\} \\
\mathbb{Q}^c &:= \mathbb{R} \setminus \mathbb{Q}
\end{align*}
\]

Now let’s go over some set operations and other important definitions.

**Definition 2.**

1. The union of sets \( A \) and \( B \) is \( A \cup B := \{x : x \in A \text{ or } B\} \).
2. The intersection of sets \( A \) and \( B \) is \( A \cap B := \{x : x \in A \text{ and } B\} \).
3. The complement (set minus) of \( B \) relative to \( A \) is \( A \setminus B := \{x : x \in A \text{ and } x \notin B\} \).

**Definition 3.** The empty set, denoted as \( \emptyset \), is a set with no elements.

**Definition 4.** Two sets \( A \) and \( B \) are said to be disjoint if \( A \cap B = \emptyset \).

We went over examples of these definitions in class.

Now let’s look at a theorem on set identities.

**Theorem 1.** Consider the sets \( A, B, C \). The following identities hold,

1. \( A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C) \)
2. \( A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C) \)

Before we prove the theorem, let’s first think of what strategy we are going to use. Recall that in order to prove equality we have to do two proofs each:

1. (a) \( A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C) \)
2. (b) \( (A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C) \)
3. (a) \( A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C) \)
4. (b) \( (A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C) \)

**Proof.**

(1) (a) Suppose \( x \in A \setminus (B \cup C) \), then \( x \in A \), but \( x \notin B \cup C \Rightarrow x \notin B, x \notin C \Rightarrow x \in A \setminus B \text{ and } x \notin A \setminus C \Rightarrow x \in A \setminus B \cap (A \setminus C) \).

(b) Suppose \( x \in (A \setminus B) \cap (A \setminus C) \), then \( x \in A \setminus B \text{ and } x \in A \setminus C \Rightarrow x \in A, x \notin B, x \notin C \Rightarrow x \notin (B \cup C) \Rightarrow x \in A \setminus (B \cup C) \).

Since (a) and (b) hold, \( A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C) \).

(2) (a) Suppose \( x \in A \setminus (B \cap C) \), then \( x \in A \), but \( x \notin (B \cap C) \Rightarrow x \notin B \text{ or } x \notin C \). If \( x \notin B \), then \( x \in (A \setminus B) \). If \( x \in C \), then \( x \in (A \setminus C) \Rightarrow x \in (A \setminus B) \cup (A \setminus C) \).

(b) Suppose \( x \in (A \setminus B) \cup (A \setminus C) \), then \( x \in (A \setminus B) \text{ or } x \in (A \setminus C) \). If \( x \in (A \setminus B) \), \( x \in A \) but \( x \notin B \). If \( x \in (A \setminus C) \), \( x \in A \) but \( x \notin C \Rightarrow x \notin (B \cup C) \Rightarrow x \in A \setminus (B \cap C) \).

Since (a) and (b) hold, \( A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C) \).

Thereby completing the proof. \(\square\)