6.2 The Mean Value Theorem

Here we will just cover a bunch of theorems that will lead up to the Mean Value Theorem.

**Theorem 1.** If $f$ defined on $(a,b)$ has a local maximum (or minimum) at $x$, and $f$ is differentiable at $x$, then $f'(x) = 0$.

*Proof.* Since $f(x)$ is maximum at $x$, $f(x) ≥ f(x+h)$ for all $h$ such that $x+h ∈ (a,b)$. Then $f(x+h)−f(x) ≤ 0$. If $h ≥ 0$,

$$\lim_{h→0^+} \frac{f(x+h)−f(x)}{h} ≤ 0,$$

and if $h < 0$,

$$\lim_{h→0^-} \frac{f(x+h)−f(x)}{h} ≥ 0.$$  

Since the derivative exists, $f'(x) = 0$ because otherwise the left hand derivative and right hand derivative would be different. □

**Theorem 2** (Rolle’s). If $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$, and $f(a) = f(b)$, then there is an $x ∈ (a,b)$ such that $f'(x) = 0$.

*Proof.* Since $f$ is continuous, a maximum and minimum exist. If the maximum or minimum occurs in the interior, $f'(x) = 0$ by the previous theorem. If they occur at the end points, $f(x) = f(a) = f(b)$, so it is a constant, and therefore the derivative is trivially $f'(x) = 0$. □

**Theorem 3** (Mean Value Theorem). If $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$, then there is a $ξ ∈ (a,b)$ such that

$$f'(ξ) = \frac{f(b)−f(a)}{b−a}. \quad (1)$$

*Proof.* Let’s define a function

$$h(x) := f(x) − \frac{f(b)−f(a)}{b−a}[x−a]. \quad (2)$$

Notice that $h$ satisfies the hypotheses of Rolle’s theorem. Then $h'(ξ) = 0$ for some $ξ ∈ (a,b)$, and hence

$$f'(ξ)−\frac{f(b)−f(a)}{b−a} = 0 \Rightarrow f'(ξ) = \frac{f(b)−f(a)}{b−a}.$$  

□

6.3 Indeterminate Forms

We will just go over this briefly since you have seen all of this in Calc I.

Recall the types of indeterminate forms

$$\begin{array}{cccccc}
0 & 0 & \infty & \infty & 0 \cdot \infty & \infty−\infty & 0^0 & 1^\infty & \infty^0.
\end{array}$$

Remember that L'Hôpital can only be used with the first two cases, which means you would need to convert any other case to the type in the first two: $0/0$ or $∞/∞$.

Let’s look at a couple of examples,

$$\lim_{x→∞} 1^x = 1$$

because the base is already unity. It is not changing. So if we take $x$ as big as we want $1^x$ will still be 1.
Now let's look at a $1^\infty$ case that is actually indeterminate,

$$ L = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x. $$

For this one we need to use our $e^{\ln}$ trick.

$$ L = \exp \left( \lim_{x \to \infty} x \ln \left(1 + \frac{1}{x}\right) \right). $$

We need to look at the argument separately, and then plug it back in if it exists. Notice that the argument, however, is not in a proper indeterminate form. We need to change it to one of the two cases where we can use L'Hôpital.

$$ \lim_{x \to \infty} x \ln \left(1 + \frac{1}{x}\right) = \lim_{x \to \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{1/x}. $$

Then applying L'Hôpital give us

$$ \lim_{x \to \infty} \frac{(1/x)^{n}/ \ln (1 + 1/x)}{(1/x)^n} = \lim_{x \to \infty} \frac{1}{\ln (1 + 1/x)} = 1. $$

Plugging back into the original limit gives us

$$ L = \exp \left( \lim_{x \to \infty} x \ln \left(1 + \frac{1}{x}\right) \right) = e $$

### 6.4 Taylor’s Theorem

Suppose the function $f$ has the following power series:

$$ f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n. \hspace{1cm} (3) $$

Can we figure out what the coefficients are? Yes, yes we can. Notice that $f(a) = c_0$, so that gives us the first coefficient. For the second one let's differentiate to get $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$. Now, if we plug in $a$ we get $f'(a) = c_1$. How about the third? Well, $f''(x) = 2c_2 + 6c_3(x-a) + \cdots$, so $f''(a) = 2c_2$. Can we figure out what $c_n$ should be? Well we see that if we keep taking derivatives and evaluating them at the center, we get $f^{(n)}(x) = n!c_n + \cdots$, so $c_n = f^{(n)}(x)/n!$. We have just derived a general formula for finding the coefficients of our series.

**Definition 1.** The series representation

$$ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \hspace{1cm} (4) $$

is called a Taylor series of $f$ at $x = a$. If $a = 0$ we simply call this the Taylor series of $f$ at $x = 0$ or the McLaurin series of $f$ - both are used interchangeably.

**Theorem 4 (Taylor).** Let $f : [a, b] \to \mathbb{R}$ have $n$ continuous derivatives and let $f^{(n+1)}$ exist on $(a, b)$. Then for $x_0 \in [a, b]$,

$$ f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \hspace{1cm} (5) $$

for some $\xi$ between $x$ and $x_0$. 
Proof. For some $t$ between $x$ and $x_0$ define $F$ as
\[ F(t) := f(x) - f(t) - (x-t)^n f'(t) - \cdots - \frac{(x-t)^n}{n!} f^{(n)}(t). \] (6)
Taking the derivative gives us
\[ F'(t) = -f'(t) + f'(t) - (x-t)f''(t) + \frac{2}{2!} (x-t)f'''(t) - \cdots - \frac{(x-t)^{n-1}}{(n-1)!} f^{(n-1)}(t) + \frac{n}{n!} (x-t)^n f^{(n)}(t) - \frac{(x-t)^n}{n!} f^{(n+1)}(t). \]
Now define
\[ G(t) := F(t) - \left( \frac{x-t}{x-x_0} \right)^{n+1} F(x_0), \] (7)
then $G(x_0) = G(x)$. By the Mean Value Theorem, there is a $\xi$ between $x$ and $x_0$ such that $G'(\xi) = 0$.

Therefore,
\[ F(x_0) = -\frac{1}{n+1} \frac{(x-x_0)^{n+1}}{(x-\xi)^n} F'(\xi) = -\frac{1}{n+1} \frac{(x-x_0)^{n+1}}{(x-\xi)^n} \frac{(x-t)^n}{n!} f^{(n+1)}(t) = \frac{(x-t)^n}{n!} f^{(n+1)}(\xi) (x-x_0)^{n+1}. \]

Notice that this is precisely the remainder of the Taylor series. \qed

Ex: Find the Taylor series of $f(x) = e^x$ and its radius of convergence.

Solution: This is easy because we can find the $n$th derivative of $e^x$ straightaway, i.e. $f^{(n)}(x) = e^x$, hence $f^{(n)}(0) = 1$. So $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Now, this is still a power series so like any other power series we can find the radius of convergence by using either root or ratio test. Lets apply ratio test,
\[ \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1}. \]
Taking the limit gives us $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0$, so $R = \infty$. Therefore, the Taylor series converges everywhere and it is an exact representation of $e^x$.

Ex: Find the Taylor series of $f(x) = \sin x$.

Solution: Again we have a nice pattern for this one (Hint: I like functions with nice patterns!) $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, and the pattern just keeps repeating, so
\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \]