My interests are in formulating simple models that agree with experiments of real-world systems and analyzing them via rigorous mathematics. Through my expertise in dynamical modeling and analysis via dynamical systems theory, asymptotic methods, and bifurcation theory, my research program has focused on cancer drug response, walking droplets, and chaotic logical circuits. I believe attacking a problem from different angles through models of varying complexity often provides insights and information that prove crucial in solving or making progress in the solution of the problem. In order to accomplish this, I develop and analyze simple mathematically tractable models that preserve significant qualitative features and possibly some quantitative behavior of physical phenomena.

I. CANCER DRUG RESPONSE

I.A. Background. According to the National Cancer Institute, almost 40% of men and women in the United States end up developing cancer in their lifetime, and total national expenditure on cancer is 125 Billion dollars. While we cannot hope to eradicate cancer in the near future, early detection and treatment can decrease death rates. However, keeping a human being alive is not the main goal; it is to maintain their and their family’s quality of life. This can only be achieved through individualized treatment which increase efficacy and decrease toxicity.

In recent years there has been a lot of effort in predictive modeling through data analysis. While these techniques are effective and important, they only reveal a correlation, but not a fundamental cause and effect. On the dynamics side of cancer research, most of the models are population models, which are standard in many biological phenomena. However, there is a dearth of biophysical models, which are necessary to understand the fundamental interactions of the drugs and the cancer. My current work, in collaboration with Souparno Ghosh, endeavors to rectify this gap.

I.B. Current Work. Once again, borrowing statistics from the National Cancer Institute, over 60% of cancer cases and over 70% of cancer related deaths occur in Africa, Asia, and South America, having the worst effect on the poorest populations. Some of the easiest cancers to treat in the Western world, solid – accessible tumors, are often fatal in poor nations. In industrialized nations the answers to solid tumors is simple – operate. However, operation is generally quite complex, and costs a significant amount of money. In the seminal work of Sugiura et al. [1] it was shown that ethanol injections can successfully deteriorate malignant tissue. Recently, Morhard et al. [2] developed a new method employing ethyl cellulose in the injection for which single-phase small-volume treatments suffice to trigger ablation. These forms of treatment are much cheaper than surgery and can be
administered by anyone, including local doctors. If this can be done with ethanol, perhaps similar procedures can be conducted with other drugs.

I have been modeling the fluidic interactions of the drug with the geometry and topography of the tumor. While I have developed a hierarchy of more than half a dozen models, there is one promising simple model we are focusing on for mathematical and statistical analysis: drug diffusion through a uniform spherical tumor with a leaky boundary and constant oxygen concentration. Since our initial and boundary conditions do not allow for simple analytical solutions, I developed finite difference (FTCS, BTCS, C–N) simulations of the model, each scheme serving a different purpose. A snapshot of one of the numerical simulations is shown in Fig. 1.

The main goal of this form of modeling is to optimize the parameters to make predictions on populations and use physical parameter data to develop a treatment strategy for an individual patient. This requires extraction of the same forms of data from the model as is observed in experiments and clinical trials. One way efficacy is recorded is in the form of dose-response curves. These are produced by administering various levels of doses to tumors and measuring the amount of tumor cells that die off after specified periods of time. From the model I produce both dose-response curves for specified periods of time and does-response surfaces, which give us a fine-grain picture of the efficacy of the treatment. Both the curves and surface are illustrated in Fig. 2.

II. WALKING DROPLETS

II.A. Background. It has been known for decades that, given proper conditions, a fluid drop can be made to bounce on a vibrating fluid bath for long time scales. In recent years, bouncing droplets have been observed to bifurcate from the bouncing state (no horizontal motion) to the walking state (horizontal motion). Experiments with walking droplets (called walkers) exhibit analogs of wave-particle duality and more specifically quantum-like phenomena. Studying walkers can enhance our understanding of quantum mechanics and suggest viable alternatives to the Copenhagen interpretation.

The seminal work of Couder, Fort, and coworkers [3] showed a bouncing droplet bifurcating to the walking. This led to many more experiments and efforts to model the observed dynamics. While the (integro-differential) models, for free space and rotating frame, developed for walkers agree well with the experiments [4, 5], they are quite complex and difficult to analyze. This necessitated simpler (often discrete) dynamical models.

II.B. Current Work. My research has focused on attacking walking dynamics on two fronts: developing simple models for various geometries and analyzing well established models via dynamical systems theory. Recently, this has manifested in modeling multiple non-chaotic walkers and single chaotic walkers in an annulus, and analyzing a novel bifurcation arising in a model by Gilet [6].

For multiple drops in an annulus, an experimental study was conducted showing that the addition of a drop near the original set of drops increases their collective speed [7]. The timescale of going from one collective speed to the next is so fast that it is observed as being instantaneous in experiments. However, one may intuit that this does not occur instantaneously. I modeled the effect as...
an iterated map that receives a “kick” once a new droplet is added, similar to a kicked rotator. The collective velocity then rises until it reaches its new steady-state. A natural test for the model is to compare the sequential steady-state velocities (green stars) with that of experiments (red markers with error bars) as shown in Fig. 3a.

The added energy provided by the additional droplet increases the slope of the impact cavities, which in turn increases the velocity in a system far below the chaotic threshold. Therefore, one might ask what happens when there is a single drop on an annulus with the bath being vibrated at or above threshold. I model the walking as a damped kicked rotator, similar to the multi-drop system, but with the assumption that the single drop feels its own kick and this kick can either propel or retard the droplet. Mapping the iterates of this model admits the chaotic attractor in Fig. 3b.

On the theoretical front, under the supervision of Denis Blackmore, I discovered a new global bifurcation leading to chaos in Gilet’s discrete dynamical model for walking in a semi-confined geometry, where, $w$ is the amplitude of the wave, $x$ is the position of the walker, $C \in [0, 1]$ is a constant representing wave-particle coupling, $\mu \in [0, 1]$ is the damping factor, and $\Psi \in \mathbb{R}$ is a single eigenmode of the Faraday wave field. To derive the model, Gilet defines the damping factor as the ratio of the wave amplitude just before the next impact ($w_{n+1}$) to the wave amplitude just after the previous impact ($w_n + \Psi(x_n)$). It is then assumed the difference in horizontal position of the droplet is proportional to the gradient of the wave field at impact $n$.

Since the waves are 1-D, the wave field gradient is $w_n \Psi'(x_n)$. As we can observe, the direction of motion will always be opposite to the sign of the gradient and the proportionality constant determines how much the droplet is affected by the wave, thereby deriving (1)).

Denis Blackmore and I proved that the global bifurcation occurs [8] after an invariant circle (Fig. 5a) from a Neimark–Sacker bifurcation grows towards a saddle as the bifurcation parameter ($\mu$ or $C$) increases. The unstable manifold of the saddle meets the stable manifold tangentially at one point and subsequently intersects transversely at two points. This causes a “blinking” effect on the invariant circle and later on the chaotic attractor (Fig. 5b).
II.C. Previous Work. In [9], I proved the existence of (N-S) for both $\mu$ and $C$ and conjectured the existence of a new global bifurcation mentioned in the previous section. My theorems accurately predict the existence of supercritical (N-S) for $\mu$ as observed in Gilet’s simulations. In addition, the theorems predict the existence of both supercritical (N-S) for $C$ and subcritical (N-S) for $\mu$ and $C$, which were not observed in the original simulations. I also produced numerous simulations that show complete agreement between the theory and numerics.

The theorems prove the existence of (N-S) in $\mu$ by holding $C$ constant and in $C$ by holding $\mu$ constant. In Fig. 6a, the green lines represent the local stable manifold and the red lines represent the local unstable manifold at the respective saddle fixed points. We observe from the figure that at $\mu = 0.79$ there exists an invariant circle that arises from the supercritical (N-S). For the eigenmodes chosen in this simulation, as $\mu$ is increased each relevant fixed point bifurcates into invariant circles which increase in radius until getting arbitrarily close to the stable manifold. Once past this value of $\mu$ the next fixed point bifurcates. A similar bifurcation behavior occurs when varying $C$ while holding $\mu$ constant.

Furthermore, the theorem leads us to a subcritical (N-S), which the original simulations missed. In Fig. 6b the blue iterates have an initial point inside the invariant circle, and the red iterates have an initial point outside the invariant circle. Finally, the black dashed curve represents the unstable invariant circle. While relevant fixed points for most eigenmodes will undergo supercritical (N-S), the theorem proves the existence of eigenmodes for which some fixed points undergo subcritical (N-S).

III. Nonlinear Logical Circuits

III.A. Background. Logical circuits are an integral part of modern life that are traditionally designed with minimal uncertainty. While this is straightforward to achieve with electronic logic, other logic families such as fluidic, chemical, and biological circuits naturally exhibit uncertainties due to inherent nonlinearity. In addition, chaotic logical circuits have the potential to be employed in random number generation, encryption, and fault tolerance. However, in order to exploit the properties of various nonlinear circuits they need to be studied further. Since experiments with large systems become difficult, tractable mathematical models that are amenable to analysis via dynamical systems theory are of particular value.

III.B. Current Work. Our construction (Fig. 7) of the chaotic logical circuits, developed by Ian Jordan through the NJIT Phase I and II Provost undergraduate research grants under my supervision, mainly employ the usual circuit elements such as resistors, capacitors, and inductors, and a less common component called a nonlinear resistor. The most well-known nonlinear resistor is Chua’s diode, which is used as an integral element in Chua’s circuit [10, 11]. Simple chaotic flip-flops use
a small number of these components, however for more complex logical circuits, we use threshold control units (TCUs). One such circuit that exhibits interesting dynamical behavior is the chaotic Set-Reset flip-flop (RSFF). Fig. 7 shows our physical realization of a chaotic NOR – RSFF and Fig. 8 shows a “black box” schematic of a NOR – RSFF.

In addition to supervising the construction and developing the experiments for the RSFF, I have developed robust models for the TCUs and both deterministic and stochastic models for the chaotic RSFF and NOR gates [12]. First, I modeled the TCU dynamics as combination of tent maps due to the piecewise-linear manner in which operational amplifiers affect incoming voltages. Then, I modeled the RSFF/NOR as continuous extensions of the standard boolean outputs, and later modeled certain voltage changes as stochastic processes due to the unpredictability of the outputs during those changes.

As shown in Fig. 9, the simulations of the models agree with the qualitative behavior observed in experiments [12]. This is further solidified by two of my theorems that prove the observed dynamics are in fact chaotic, and not just periodic with high period; that is, if we kept the set/reset inputs at unity, the unpredictable behavior would continue ad-infinitum as in the physical circuit (as long as it is powered).

Finally, an example of encryption through an RSFF that I developed as a proof of concept MATLAB code using this model [13] is shown in Fig. 10. Here Encryptor 1 and Decryptor 1

```
Command Window

Prerelease License — for engineering feedback and testing purposes only. Not for sale.

>> Encryptor1('Thesis.txt','SecretMessage.txt')
>> type SecretMessage.txt

GFONJ51PTG7AVGPPWBLDAJUKCWXZOUWMTJYMBBZBSCXJ0XZLFDXOU WSNBDPVMCEYIQCT WFTBLOZDOCICKFPRKRYEYP
>> Decryptor2('SecretMessage.txt','OriginalMessage.txt')
>> type OriginalMessage.txt

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>> Decryptor3('SecretMessage.txt','OriginalMessage.txt')
>> type OriginalMessage.txt

SPPXW RZNPWPMHG PHZTP2YDZLMHR GHZIMUXKJH错误MPTAGIAIPQJ GQFENMQTXPSHVMACPHZBQIALM KTMROBZDUJFQ
```

F i g u r e 9. Simulations of TCUs and NOR gates

F i g u r e 10. Example of chaotic encryption implemented in MATLAB
uses the first threshold, and is employed by “Person 1”. Similarly, Encryptor 2 and Decryptor 2 uses the second threshold, and is employed by “Person 2”. At each chaotic cycle of the thresholds the Encryptor/Decryptor pairs integrates over their respective voltages, which provides the new encryption code. Since users only have access to one threshold they will not be able to decrypt their own message as shown in the last line of Fig. 10.

III.C. Previous Work. As an undergraduate, I numerically simulated a discrete dynamical model of the RSFF [14], for the case \((R, S) = (1, 1)\), the region in which the interesting behavior arises in both chaotic and standard RSFFs. From the simulations we observed and later proved the existence of behavior such as Neimark-Sacker bifurcations and horseshoe chaos.

Early in my dissertation research, I had developed a variety of models by examining gates and TCUs. This technique gives the models more flexibility compared to an ad hoc model — they can be modified slightly to capture dynamics arising in other systems in addition to the ones being studied. One promising model (2) captures the effects of the case \((R, S) = (1, 1)\) similar to the model in [15],

\[
\xi(x, y) := \frac{y(1-f(x))}{1-(1-f(x))(1-g(y))}, \quad \eta(x, y) := \frac{x(1-g(y))}{1-(1-f(x))(1-g(y))}. \tag{2}
\]

In this model, \(x\) and \(y\) are the iterated threshold inputs, \(\xi\) and \(\eta\) are the respective outputs, and \(f\) and \(g\) are functions describing the effects the components have on the incoming signal. Here, for certain functions of \(f\) and \(g\), various types of chaotic dynamics have been observed and proved.

IV. Dynamical Systems Theory

In addition to applying dynamical systems techniques to physical and biological problems, I am also interested in dynamical systems theory in itself.

IV.A. \(\sigma\)-map Bifurcations. After discovering the new global bifurcation (Fig. 5b) in Gilet’s model (1), we developed a generic dynamical system in order to generalize the theory of such bifurcations [16]. We dubbed this map, the \(\sigma\)-map due to the geometry of the set of interest. The progression of the topology of the map transitioning from regular dynamics to chaotic dynamics is shown in Fig. 11. The transition from tangential to transverse intersections of the unstable and stable manifolds is

\[\text{Figure 11. Progression of the unstable manifold as the bifurcation parameter is increased.}\]
what causes the “blinking” of the attractor, which eventually leads to chaotic dynamics. An example of a planar map containing $\sigma$-type dynamics is,

$$F(x, y, \lambda) := \left( x - \lambda \psi'(x)y, \frac{1}{10}(y + \psi(x)) \right); \quad \psi(x) := \frac{216}{11} x \left( \frac{x^2}{3} - \frac{9}{8} x + \frac{5}{4} \right).$$

(3)

where $\lambda$ is our bifurcation parameter.

IV.B. Generalized Attracting Horseshoe. In [17], we showed that attracting horseshoes may be generalized to be contained within a quadrilateral trapping region. Through the NJIT Provost high school internship and the Provost Phase 1 undergraduate research grant, I mentored Karthik Murthy (Bridgewater-Raritan High School) and Parth Sojitra (NJIT-ECE), under the supervision of Denis Blackmore, in finding numerical evidence of generalized attracting horseshoes (GAH) in Poincaré maps of flows that can be used to model circuits. One such system of ODEs was the Rössler attractor. I developed the algorithms to go from a first return map to a Poincaré map and finally to find the quadrilateral trapping region for the supposed GAH. I then guided our students in writing the MATLAB codes to carry out the algorithms. While finding the trapping region numerically (Fig. 12) is not a proof, it does give us confidence that there exists a GAH in the Poincaré map of the Rössler attractor.

V. Future Work

Currently, I have been focusing my efforts on cancer drug response and walking droplets because of the abundance of both physical and mathematical problems, and I anticipate on continuing with this focus. However, when the opportunity arises, I will apply my work on electronic logical circuits to other logic families. Furthermore, I have a broad range of scientific interests and would be very much open to study phenomena from other fields such as ecology, neuroscience, economics, social science, etc.

V.A. Cancer Drug Response. Through my hierarchy of models, I have given myself an immense amount of future work. The current model assumes constant oxygen concentration throughout the tumor and it also only considers efficacy, but not toxicity. Furthermore, there are more complex fluid processes and tumor geometries that would significantly alter the dynamics. As I analyze more complex models, I hope to predict previously unobserved phenomena and develop techniques for individualized treatments.

In addition, I am interested in developing new models for physical experiments to help reduce costs and increase the chances of discovery. One such experiment, proposed by Dimitri Pappas and Akif Ibragimov, endeavors to analyze many treatment strategies at one time. Essentially, the idea is to parallelize the testing process which would both decrease time to find an effective treatment and decrease costs. In the near future, we have plans for Akif and me to analyze Akif’s models and develop new models.

Finally, one of the major goals of my collaboration with Souparno Ghosh is to use both statistical and mathematical analysis to develop scientifically rigorous models with high predictive capabilities. In the near future, we look to explore modeling cancer treatments via principle differential analysis.
V.B. **Walking Droplets.** Due to the nascency of the field and complexity of the phenomena of interest, there is an abundance of novel problems to pursue. My goal is to both push the boundary of dynamical systems theory applied to existing models and to develop new models to better understand the quantum-like experiments.

In addition to developing models for walking on an annulus, I have begun developing models for walking on an ellipse, which interestingly yields symbolic dynamics in the choice of eigenmodes. This implies that there are some very simple dynamics overlayed on top of the complex dynamics. Furthermore, I have also begun work on analyzing a simplified three ODE model of walking in free space developed by Luiz Faria.

From the theoretical side, in my study of (N-S) in Gilet’s model I observed crises bifurcations of the chaotic attractors, and I have been investigating them numerically to better understand their dynamics, which should prove useful in the very difficult task of analyzing them.

Finally, a pertinent goal of those in the field is to tune the experiments such that the probability density function observed for the location of the droplet is equivalent to that of the time-dependent Schrödinger equation. This can be accomplished by making predictions based on invariant measures of the chaotic attractors, which I plan to undertake in the near future.

V.C. **Logical Circuits.** There are two main motivating factors for studying chaotic logical circuits: to learn how other more complex logic families carry out logical operations and to exploit the combination of chaotic and logical behavior to develop new technologies. One application that has been discussed and attempted is chaotic encryption. While researchers have not been successful thus far, I believe using multiple RSFFs can remedy the problems past investigations have encountered.

Although electronic logical circuits have been exhaustively studied, there are other logic families that are in need of mathematical modeling. In the near future, I look to model biological, microfluidic, and/or ferrofluidic logic families, which I would then study using techniques we developed for electronic logical circuits. I will design circuits, such as the RS flip-flop, in these systems and modify our models to capture the dynamics of the new logic families. This would reveal ways in which biological and other systems carry out arithmetic operations and may consequently give us a glimpse of how these systems evolved to become “intelligent”.
REFERENCES


