Locating Roots (and Poles) :
Rouché’s & Hurwitz’s Theorems

Josh Engwer
Texas Tech University

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NOTATION: \( \mathbb{N} = \{1, 2, 3, \ldots\} \)

\( B(a; r) \equiv \{z \in \mathbb{C} : |z - a| < r\} \quad \bar{B}(a; r) \equiv \{z \in \mathbb{C} : |z - a| < r\} \quad \partial B(a; r) \equiv \{z \in \mathbb{C} : |z - a| = r\} \)

\( \gamma \in \mathcal{CRC}(G) \equiv \gamma \) is a closed, rectifiable curve in open set \( G \)

\( n(\gamma; a) := \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - a} \, dz \equiv \text{index (or winding number)} \) of \( \gamma \) w.r.t. the point \( z = a \) \hspace{1em} \text{[provided} \ a \not\in \gamma \text{]} \)

\( \gamma \approx 0 \equiv \gamma \) is homologous to zero \( \iff n(\gamma; w) = 0 \ \forall w \in \mathbb{C} \setminus G \)

\( H(G) \equiv \{ \text{analytic functions on region } G \} \quad M(G) \equiv \{ \text{meromorphic functions (analytic except for poles) on } G \} \)

\[ \{f_n\} \xrightarrow{\text{gakT}} f \equiv \{f_n\} \xrightarrow{\text{uniformly on compact subsets}} f \quad \text{”gakT” stands for ”gleichmäßig auf kompakten Teilmengen”} \]

**ARGUMENT PRINCIPLE:** \( \left[ \int_{\gamma} \frac{f'(z)}{f(z)} \, dz \right. \) is interpreted as the total change in \( \arg f(z) \) as \( z \) traces the curve \( \gamma \)

Let \( G \) be an open set and \( f \in M(G) \) with poles \( p_1, \ldots, p_m \) and zeros \( z_1, \ldots, z_n \) counted by multiplicity.
If \( \gamma \in \mathcal{CRC}(G) \) with \( \gamma \approx 0 \) and passing thru none of the poles \( p_1, \ldots, p_m \) or zeros \( z_1, \ldots, z_n \),

then, \( \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j) \)

**ROUCHÉ’S THEOREM:**

- (Roots & Poles) Let \( \epsilon > 0 \) and \( f, g \in M(B(a; r + \epsilon)) \) with no zeros or poles on the circle \( \partial B(a; r) \).

  Let \( Z_f, P_f, Z_g, P_g \) denote the number of zeros and poles of \( f, g \) inside \( \partial B(a; r) \) counted by their multiplicities.

  Then \( |f(z) + g(z)| < |f(z)| + |g(z)| \ \forall z \in \partial B(a; r) \implies Z_f - P_f = Z_g - P_g \)

- (Roots only) Let \( \epsilon > 0 \) and \( f, g \in H(B(a; r + \epsilon)) \) with no zeros on the circle \( \partial B(a; r) \).

  Let \( Z_f, Z_g \) denote the number of zeros of \( f, g \) inside \( \partial B(a; r) \) counted according to their multiplicities.

  Then \( |f(z) + g(z)| < |f(z)| + |g(z)| \ \forall z \in \partial B(a; r) \implies Z_f = Z_g \)

**HURWITZ’S THEOREM:** Throughout, \( G \) is assumed to be a region (i.e. open connected set)

- (Theorem) Let \( \{f_n\} \subset H(G) \) s.t. \( \{f_n\} \xrightarrow{\text{gakT}} f \not\equiv 0 \) and let \( \bar{B}(a; r) \subset G \) s.t. \( f(z) \not\equiv 0 \ \forall z \in \partial B(a; r) \)

  Then \( \exists N \in \mathbb{N} \) s.t. \( n \geq N \implies f \) and \( f_n \) have the same number of zeros in \( B(a; r) \)

- (Corollary) Let \( \{f_n\} \subset H(G) \) s.t. each \( f_n \) has no zeros in \( G \).

  Then \( \{f_n\} \xrightarrow{\text{gakT}} f \in H(G) \implies \text{either } f \text{ has no zeros in } G \ \text{ OR } f \equiv 0 \)
References