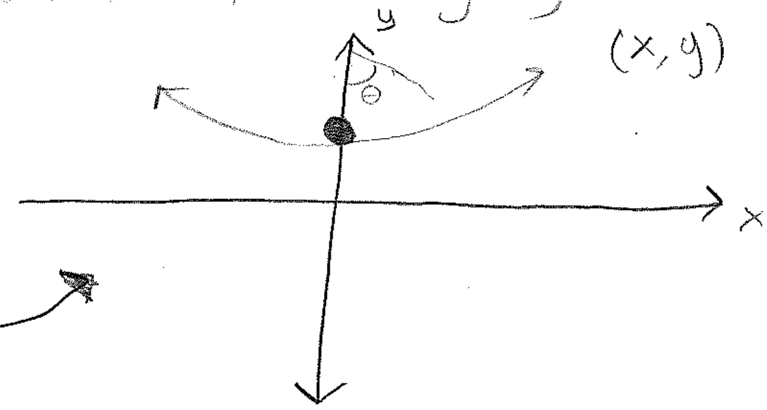
Quantum Homotopy

Stephen Peña - Mechanics on Manifolds and Classical Field Theory

(Monday, September 9, 2019)

$\boxed{\text{Outline}}$ - 1) Classical Mechanics
2) Quantum Mechanics
3) Gauge Theory
4) Functorial Quatum Field Theory (FQFT), Factorization Algebras

Mechanics is dominated by Hamilton's Principle of Least Action: We discuss generalized coordinates. $\to q^I$ Lets look at a pendulumn swinging in x-y space



$x^2+y^2=0$ is constraint

Def: Constraints are equations your system must obey.

Def: A motion in $\mathbb{R}^n$ is a differentiable mapping $\vec{X}: I \longrightarrow \mathbb{R}^n$

$\dot{\vec{X}}$ is short hand for $\frac{d\vec{x}}{dt}$. Also

$\dot{x}$ = velocity

$\ddot{x}$ - acceleration

Def: A motion of a systems of n-points is given by n mappings $X_i: \mathbb{R} \longrightarrow \mathbb{R}^N$.

Def: The direct product of n copies of $\mathbb{R}^N$ is called configuration space.

We have

$$F: \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}^N$$

such that $\ddot{x} = F(x, \dot{x}, t) \leftarrow$ (F=ma)

Def: A system with one degree of freedom is a system defined by one differential Equation:

$$\ddot{x} = f(x), \quad x \in \mathbb{R}$$

Def: The Kenetic energy is the quadratic form

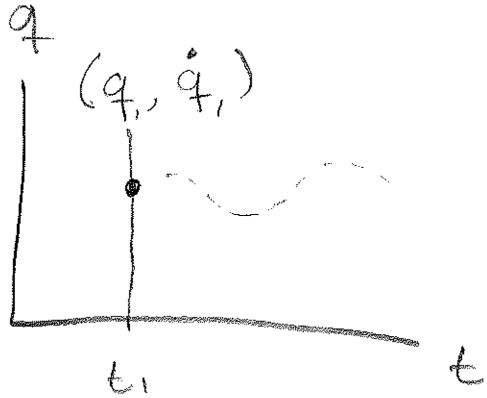
$$T = \frac{1}{2} x^2$$

Def: The Potential energy is the function

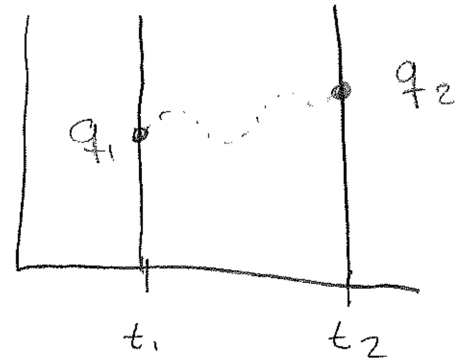
$$\mathcal{U}(x) = -\int_{x_0}^{x} f(\xi)\, d\xi$$

The total energy is $E = T + \mathcal{U}$.

Two ways to view Motion



IVP



"boundary" value

$\boxed{\text{Hamilton's Principle}}$: Motion of a system in a time interval $t \in (t_1, t_2)$ Coincides with the extrenum of the action functional.

$$S = S[q] = \int_{t_1}^{t_2} L\,(q, \dot{q}, t)\, dt$$

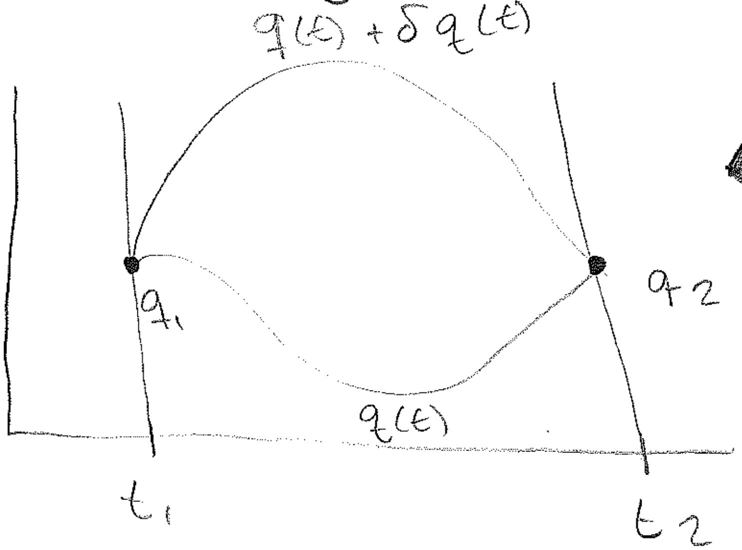
Miricle Function

"Lagrangien"

How do we get it? Calculus of Variation.

Let $X = (\mathbb{R}^N)^I$ $\left(= \{\text{curves in space}\}\right)$

Consider again



we will vary curves

$\delta q(t)$ vanishes at t_1, t_2.

$$* \quad \delta \frac{d}{dt} = \frac{d}{dt} \delta$$

So

$$\delta S = \delta \int_{t_1}^{t_2} L \, dt$$

$$= \int_{t_1}^{t_2} \delta L \, dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial q} \delta \dot{q} \right) dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q \right) dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \delta q \right) dt$$

$$\overset{\text{by parts}}{=} \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q \, dt + \underbrace{\cancel{\left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2}}}_{=0}$$

Since $\delta q \neq 0$ we have

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0 \quad (*)$$

Euler-Lagrange Equations

Remarks: 1) 2^{nd} order ODE

2) $L = T - V$

3) $L'(q, \dot{q}, t) = L + \frac{d\lambda(q, t)}{dt}$

To solve (*) we have the following ways

i) Analytically (Integrable Systems)

ii) Pertubations

iii) Numerical

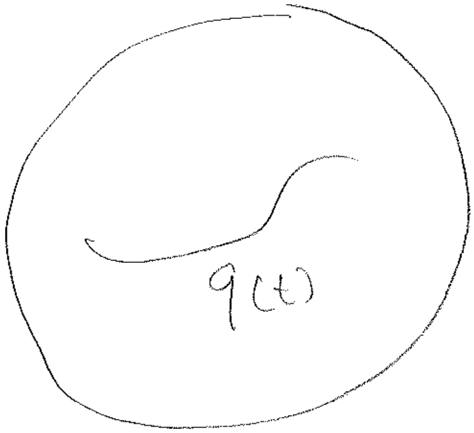
Hamiltonian Mechanics :

Why another formalism?

Trade E-L $\longrightarrow$ Hamiltons Equations

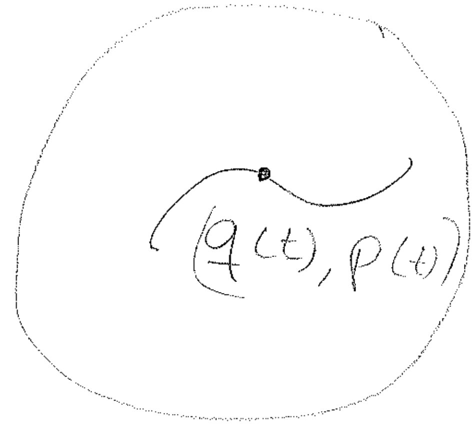
n - 2^{nd} order $\qquad$ 2n - 1^{st} order

LM



(t, q(t))

HM Phase space



$\{q^i, P_i\}$

where $P_i := \frac{\partial L}{\partial \dot{q}}$.

Hamitonian = Legendre transform of Lagragian

$$H(q, P, t) = P_i \dot{q}^i - L$$

$\boxed{\text{Hamilton's Theorem}}$: Systems of n E-L is equivalent to 2n 1st order hamiltons equations

$$P_i = -\frac{\partial H}{\partial q^i} \qquad q^i = \frac{\partial H}{\partial P_i}$$

Similar process to E-L we (5)

have

$$0 = \delta S = \int_{t_1}^{t_2} [\delta p(\dot{q} - \frac{\partial H}{\partial p}) + \delta q(-\frac{\partial H}{\partial q} - \dot{p})] dt$$

Def: Let f, g be functions on phase space, then their poisson bracket is a new phase space function

$$\{f, g\} = \sum_{p_i, q^i} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right)$$

$$\{q^i, q^j\} = 0 = \{p_i, p_j\}$$

$$\{q^i, p_j\} = \delta_{ij}$$